SPECTRAL TRANSFORMS IN THE CYCLE 43 OF ARPEGE/IFS.

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Abstract:

ARPEGE/IFS and ALADIN are spectral models, so this documentation has for purpose to describe the spectral transforms done. The present note has for aim to give a brief summary of the spectral method (how to compute Legendre polynomials, spectral transforms) and to describe parts of the code performing spectral transforms (organigramme).

Résumé:

ARPEGE/IFS et ALADIN sont des modèles spectraux, en conséquence cette documentation a pour but de décrire les transformées spectrales nécessaires pour passer de l'espace spectral vers l'espace point de grille. On y fait un bref rappel de la méthode spectrale (comment calculer les polynômes de Legendre et les transformées spectrales). On y décrit les parties de code faisant des transformées spectrales, avec fourniture de quelques organigrammes.

Contents

1   Introduction.

ARPEGE/IFS and ALADIN are spectral models. One part of computations is done in spectral space (semi-implicit scheme, horizontal diffusion scheme for model integration), the other part in grid-point space on a grid defined by a Gaussian quadrature if global model, by the plane projection if LAM model. So it is necessary to perform spectral transforms from spectral space to grid-point space or vice-versa. The present note has for aim to give a brief summary of the spectral method and to describe parts of the code performing spectral transforms. Spectral method is not described with too many details: for more details on can report to (Rochas and Courtier, 1992, note ARPEGE nr 30, in French) or to (Temperton, 1991).

For a global spectral model, spectral transforms are a combination of a Legendre transform and a Fourier transform. A spectral limited-area model like ALADIN uses a double Fourier representation for spectral fields.

∗ **Fourier transforms for transmission coefficients:** The grid-point representation of the transmission coefficients is the same as for the model variables, but the Fourier representation is done in a smaller truncation. Fourier transforms are done for these transmission coefficients in order to minimise the storing place of these coefficients: a lot of coefficients generate a lot of memory in grid-point if all the global information is stored, so only the grid-point information for one packet of latitudes is stored after the inverse Fourier transforms. To compute these transmission coefficients a model integration with full physics is done (one or several time steps, the last option being to compute a temporal average of transmission coefficients), grid-point transmission coefficients are computed by the radiation routine, then transformed into Fourier coefficients in a small truncation. To use these transmission coefficients in the simplified physics, Fourier transmission coefficients are read on a file, and at each timestep they are transformed into grid-point fields. This part of code can work only for non lagged physics. The Fourier transform package used for transmission coefficients is not the one of the "trans" library but a special one called in the grid-point part of the model.

∗ **Fast Legendre transforms (FLT):**   They are now implemented and use specific routines. They are not extensively documented there; one can refer to section 2 of (Wedi et al., 2013).

∗ **Distributed memory:**   Some distributed memory features are now introduced in the code and will be briefly described. For convenience one uses some generic appellations.

- Expression "DM-local" for a quantity means "local to the couple of processors (*proca*,*procb*)": each processor has its own value for the quantity. Expression "DM-local computations" means that the computations are done independently in each processor on "DM-local" quantities, leading to results internal to each processor, which can be different from a processor to another one.

- Expression "DM-global" for a quantity means that it has a unique value available in all the processors. Expression "DM-global computations" means that the computations are either done in one processor, then the results are dispatched in all the processors, or the same computations are done in all the processors, leading to the same results in all the processors.

- In a routine description the mention "For distributed memory computations are DM-local" means that all calculations done by this routine are DM-local; the mention "For distributed memory computations are DM-global" means that all calculations done by this routine are DM-global; when no information is provided it means that a part of calculations are DM-local and the other part is DM-global.

- Expression "main" processor currently refers to the processor number 1: (*proca*,*procb*)=(1,1).

∗ **Modifications since cycle 42:**
- Modification in call-tree for **FFTW** package.

2 Theoretical aspects.

2.1 Notations.

- a is the Earth mean radius.
- r is the radius. It appears for deep-layer equations. In the White and Bromley formulation of deep-layer equations (option **LVERCOR**=T), r is approximated by a pseudo-radius only depending on the hydrostatic pressure. In the thin layer equations, $r = a$ everywhere.
- Θ is the latitude on the Gaussian collocation grid.
- $\mu = \sin \Theta$.
- Λ is the longitude on the Gaussian collocation grid.
- n is the total wave number, m is the zonal wave number.
- M is the mapping factor.
- **V** is the horizontal geographical wind. Its zonal component is U. Its meridian component is V. Reduced components $(U^{'}, V^{'})$ are linked to unreduced components (U, V) by relationships $U = M * U^{'}$, $V = M * V^{'}$.
- D is the unreduced divergence of horizontal wind, $D^{'}$ is the reduced divergence. D and $D^{'}$ are linked by the relationship $D = (a/r) * M^2 * D^{'}$.
- ζ is the unreduced vorticity of horizontal wind, $\zeta^{'}$ is the reduced vorticity. ζ and $\zeta^{'}$ are linked by the relationship $\zeta = (a/r) * M^2 * \zeta^{'}$.
- ∇^2 is the horizontal unreduced Laplacian operator. ∇^2 is linked to the horizontal reduced Laplacian operator $\nabla^{'2}$ by the relationship $\nabla^2 = ((a/r) * M)^2 * \nabla^{'2}$.
- x and y are coordinates on the plane projection (LAM models).

2.2 Spectral representation of a scalar field.

∗ **Spherical geometry.**
A field f can be decomposed into its spectral components according to the following expression:

$$f(\Lambda, \Theta) = \sum_{m=-\infty}^{m=\infty} \sum_{n=|m|}^{n=\infty} f_{(n,m)} P_{(n,m)}(\mu) \exp{(im\Lambda)} \tag{1}$$

$f_{(n,m)}$ are spectral coefficients. $P_{(n,m)}(\mu)$ are the first kind normalised polynomials of Legendre and are normalised by the following relationship:

$$\frac{1}{2} \int_{\mu=-1}^{\mu=1} (P_{(n,m)}(\mu))^2 d\mu = 1 \tag{2}$$

That yields the following expression for $P_{(n,m)}(\mu)$, if m is a positive integer:

$$P_{(n,m)}(\mu) = \frac{1}{2^n n!} \sqrt{(2n+1)\frac{(n-m)!}{(n+m)!}} (1-\mu^2)^{m/2} \frac{d^{n+m}}{d\mu^{n+m}} (\mu^2-1)^n \tag{3}$$

For a real scalar field, $P_{(n,-m)}(\mu)$ is set equal to $P_{(n,m)}(\mu)$, $f_{(n,-m)} = \overline{f}_{(n,m)}$, so only the real and imaginary parts of $f_{(n,m)}$ for $m \geq 0$ are computed and stored. The algorithm of computation of $P_{(n,m)}(\mu)$ is described in appendix 3.

∗ **LAM models with plane projection.**
Coordinates x and y are assumed to vary between 0 and 1 in the limited-area domain. A bi-periodic field f (assumed to match $f(1,0) = f(0,0)$ and $f(0,1) = f(0,0)$) can be decomposed into its spectral components according to the following expression:

$$f(x, y) = \sum_{m=-\infty}^{m=\infty} \sum_{n=-\infty}^{n=\infty} f_{(n,m)} \exp{(i2\pi mx)} \exp{(i2\pi my)} \tag{4}$$

$f_{(n,m)}$ are spectral coefficients.

2.3 Truncation.

∗ **Spherical geometry.**
In practical the expression of f is limited to a finite set of harmonics corresponding to $0 \le n \le N_s$ and $-n \le m \le n$. That defines a triangular truncation N_s. Fourier representation of the transmission coefficients for simplified physics is a very low truncation N_{tc} always lower than the model truncation N_s. From now on formulae and algorithms will be written with truncation N_s. So equation (1) becomes:

$$f(\Lambda, \Theta) = \sum_{m=-N_s}^{m=N_s} \sum_{n=|m|}^{n=N_s} f_{(n,m)} P_{(n,m)}(\mu) \exp(im\Lambda) \tag{5}$$

Due to the aforementioned properties of $P_{(n,m)}$, expression of f becomes for a real scalar field:

$$f(\Lambda, \Theta) = \sum_{m=0}^{m=N_s} \sum_{n=|m|}^{n=N_s} f_{(n,m)} P_{(n,m)}(\mu) \exp(im\Lambda) \tag{6}$$

Some other types of truncations exist (for example rhomboidal truncation). Such truncations will not be detailed there.

∗ **LAM models with plane projection.**
Elliptic truncation: in practical the expression of f is limited to a finite set of harmonics corresponding to: $(n/N_s)^2 + (m/N_{ms})^2 \le 1$.
It is also possible (not coded) to use a rectangular truncation: $n \le N_s$ and $m \le N_{ms}$.

2.4 Horizontal derivatives.

∗ **Meridian derivative relative to Gaussian latitude Θ in spherical geometry:** for a variable f, meridian derivative is discretised in spectral space by the following formula:

$$\left(\cos\Theta \frac{\partial f}{\partial \Theta}\right)_{(n,m)} = -(n-1)e_{(n,m)} f_{(n-1,m)} + (n+2)e_{(n+1,m)} f_{(n+1,m)} \tag{7}$$

where $e_{(0,0)} = 0$ and:

$$e_{(n,m)} = \sqrt{\frac{n^2 - m^2}{4n^2 - 1}} \tag{8}$$

∗ **Zonal derivative relative to Gaussian longitude Λ in spherical geometry:** for a variable f, zonal derivative is discretised in spectral space by the following formula:

$$\left(\frac{\partial f}{\partial \Lambda}\right)_{(n,m)} = im f_{(n,m)} \tag{9}$$

Such a derivation can be done on Fourier coefficients by a multiplication by im.

∗ **Derivatives for LAM models:** example is given for zonal derivative; for a variable f, zonal derivative is discretised in spectral space by the following formula:

$$\left(\frac{\partial f}{\partial x}\right)_{(n,m)} = i2\pi m f_{(n,m)} \tag{10}$$

∗ **Horizontal laplacian in the computational sphere for spherical geometry:** for a variable f, expression of the laplacian is:

$$\left(\nabla'^2 f\right)_{(n,m)} = \left(\frac{1}{(a\cos\Theta)^2} \frac{\partial^2 f}{\partial \Lambda^2}\right)_{(n,m)} + \left(\frac{1}{a^2 \cos\Theta} \frac{\partial\left(\cos\Theta \frac{\partial f}{\partial \Theta}\right)}{\partial \Theta}\right)_{(n,m)} \tag{11}$$

One can show (for more details see appendix 1) that its discretisation writes:

$$\left(\nabla'^2 f\right)_{(n,m)} = -\frac{n(n+1)}{a^2} f_{(n,m)} \tag{12}$$

∗ **Horizontal laplacian in the computational sphere for LAM models:** for a variable f, expression of the laplacian is:

$$\left(\nabla^{'2}f\right)_{(n,m)} = \left(\frac{\partial^2 f}{\partial x^2}\right)_{(n,m)} + \left(\frac{\partial^2 f}{\partial y^2}\right)_{(n,m)} \tag{13}$$

2.5 Spectral relationship between (divergence, vorticity) and (velocity potential, stream function).

- Relationships between: reduced divergence and velocity potential χ; reduced vorticity and stream function ψ:

$$D^{'} = \nabla^{'2}\chi \; ; \quad \zeta^{'} = \nabla^{'2}\psi$$

- These relationships can be rewritten, for each complex spectral component (spherical geometry):

$$D^{'}_{(n,m)} = -\frac{n(n+1)}{a^2}\chi_{(n,m)} \; ; \quad \zeta^{'}_{(n,m)} = -\frac{n(n+1)}{a^2}\psi_{(n,m)}$$

2.6 Spectral relationship giving wind components once known (velocity potential, stream function).

- Global models: relationship between $U^{'}$, $V^{'}$, ψ and χ:

$$(U^{'}a\cos\Theta) = \frac{\partial\chi}{\partial\Lambda} - \cos\Theta\frac{\partial\psi}{\partial\Theta} \; ; \quad (V^{'}a\cos\Theta) = \frac{\partial\psi}{\partial\Lambda} + \cos\Theta\frac{\partial\chi}{\partial\Theta}$$

the spectral discretisation is:

$$(U^{'}a\cos\Theta)_{(n,m)} = im\chi_{(n,m)} + (n-1)e_{(n,m)}\psi_{(n-1,m)} - (n+2)e_{(n+1,m)}\psi_{(n+1,m)} \tag{14}$$

$$(V^{'}a\cos\Theta)_{(n,m)} = im\psi_{(n,m)} - (n-1)e_{(n,m)}\chi_{(n-1,m)} + (n+2)e_{(n+1,m)}\chi_{(n+1,m)} \tag{15}$$

- LAM models: relationship between $U^{'}$, $V^{'}$, ψ and χ:

$$U^{'} = U^{'}_{\text{moy}} + \frac{\partial\chi}{\partial x} - \frac{\partial\psi}{\partial y} \; ; \quad V^{'} = V^{'}_{\text{moy}} + \frac{\partial\psi}{\partial x} + \frac{\partial\chi}{\partial y}$$

We need a separate storage of $U^{'}_{\text{moy}}$ and $V^{'}_{\text{moy}}$ in spectral space.

2.7 Spectral relationship giving (divergence, vorticity) once known wind components.

∗ **Global models:**
- Relationship between $D^{'}$, $U^{'}$ and $V^{'}$:

$$D^{'} = \frac{1}{a\cos\Theta}\left(\frac{\partial U^{'}}{\partial\Lambda} + \frac{\partial(V^{'}\cos\Theta)}{\partial\Theta}\right) \tag{16}$$

which can be rewritten:

$$D^{'} = \frac{1}{a^2\cos^2\Theta}\left(\frac{\partial(U^{'}a\cos\Theta)}{\partial\Lambda} + \cos\Theta\frac{\partial(V^{'}a\cos\Theta)}{\partial\Theta}\right) \tag{17}$$

- Relationship between $\zeta^{'}$, $U^{'}$ and $V^{'}$:

$$\zeta^{'} = \frac{1}{a\cos\Theta}\left(\frac{\partial V^{'}}{\partial\Lambda} - \frac{\partial(U^{'}\cos\Theta)}{\partial\Theta}\right) \tag{18}$$

which can be rewritten:

$$\zeta^{'} = \frac{1}{a^2\cos^2\Theta}\left(\frac{\partial(V^{'}a\cos\Theta)}{\partial\Lambda} - \cos\Theta\frac{\partial(U^{'}a\cos\Theta)}{\partial\Theta}\right) \tag{19}$$

- Spectral discretisations of (17) and (19) allow to retrieve easily spectral components of $D^{'}a^2\cos^2\Theta$ and $\zeta^{'}a^2\cos^2\Theta$, but not directly spectral components of $D^{'}$ and $\zeta^{'}$ (requiring inversion of a pentadiagonal matrix). In practical algorithm involved to retrieve spectral coefficients of $D^{'}$ and $\zeta^{'}$ once knowing values of wind components is slightly different (requiring a division by $a\cos\Theta$ in Fourier space), and is described in detail in (Temperton, 1991) (see equations (2.16), (2.17), (2.21) to (2.27) of Temperton's paper).

∗ **LAM models:** relationship between $D^{'}$, $\zeta^{'}$, $U^{'}$ and $V^{'}$ writes:

$$D^{'} = \frac{\partial U^{'}}{\partial x} + \frac{\partial V^{'}}{\partial y} \; ; \quad \zeta^{'} = \frac{\partial V^{'}}{\partial x} - \frac{\partial U^{'}}{\partial y}$$

2.8 Steps involved in a direct and in an inverse spectral transformation for scalar fields.

∗ **Transformation from spectral to grid point space for a scalar field or a set of scalar fields in a global model.** Transformation from spectral to grid point space needs:
- An inverse Legendre transform, for each zonal wave number m. The following quantity $F_m(\mu)$ is computed for each latitude Θ:

$$F_m(\mu) = \sum_{n=|m|}^{n=N_s} f_{(n,m)} P_{(n,m)}(\mu) \tag{20}$$

  One transforms real fields, so $F_{-m}(\mu) = \overline{F_m(\mu)}$ and only $F_m(\mu)$ for $m \geq 0$ is computed.
- An inverse Fourier transform, for each latitude Θ:

$$f(\Lambda, \Theta) = \sum_{m=-N(\Theta)}^{m=N(\Theta)} F_m(\mu) \exp(im\Lambda) \tag{21}$$

  One transforms real fields, so formula (21) becomes:

$$f(\Lambda, \Theta) = F_0(\mu) + 2 \sum_{m=1}^{m=N(\Theta)} (\mathcal{R}(F_m(\mu)) \cos(m\Lambda) - \mathcal{I}(F_m(\mu)) \sin(m\Lambda)) \tag{22}$$

  $F_0(\mu)$ is the average of f on latitude Θ and is real; $N(\Theta)$ is not always equal to N_s (due to the use of a reduced Gaussian grid). For latitude Θ one obtains $nlon(\Theta)$ grid-point values.

∗ **Transformation from spectral to grid point space for a scalar field or a set of scalar fields in a LAM model.** Transformation from spectral to grid point space needs:
- An inverse meridian Fourier transform.
- An inverse zonal Fourier transform.

∗ **Transformation from grid point to spectral space for global models.** Transformation from grid point to spectral space needs:
- A direct Fourier transform, for each latitude Θ:

$$F_m(\mu) = \frac{1}{nlon(\Theta)} \sum_{jlon=1}^{jlon=nlon(\Theta)} f(\Lambda(jlon), \Theta) \exp(-im\Lambda(jlon)) \tag{23}$$

  One transforms real fields, so formula (23) becomes for non-zero m:

$$\mathcal{R}(F_m(\mu)) = \frac{1}{nlon(\Theta)} \sum_{jlon=1}^{jlon=nlon(\Theta)} f(\Lambda(jlon), \Theta) \cos(m\Lambda(jlon)) \tag{24}$$

$$\mathcal{I}(F_m(\mu)) = -\frac{1}{nlon(\Theta)} \sum_{jlon=1}^{jlon=nlon(\Theta)} f(\Lambda(jlon), \Theta) \sin(m\Lambda(jlon)) \tag{25}$$

  and for $m=0$:

$$\mathcal{R}(F_0(\mu)) = \frac{1}{nlon(\Theta)} \sum_{jlon=1}^{jlon=nlon(\Theta)} f(\Lambda(jlon), \Theta) \tag{26}$$

$$\mathcal{I}(F_0(\mu)) = 0 \tag{27}$$

  $nlon(\Theta)$ is the number of grid-points for latitude Θ. Only $N(\Theta)$ Fourier coefficients are kept. $N(\Theta)$ is not always equal to N_s.
- A direct Legendre transform, for each zonal wave number m.

$$f_{(n,m)} = \sum_{jgl=lat1(m)}^{jgl=lat2(m)} F_m(\mu(jgl)) P_{(n,m)}(\mu(jgl)) \tag{28}$$

$lat1$ and $lat2$ satisfy to $\Theta(lat2) = -\Theta(lat1)$. $lat1(m)$ is the first northern latitude where the wave number m is represented and is not always equal to 1.

∗ **Transformation from grid point to spectral space for LAM models.** Transformation from grid point to spectral space needs:

- A direct zonal Fourier transform.
- A direct meridian Fourier transform.

∗ **Relationship between N and $lat1$ (global models):**

$$N(lat1(m) - 1) < m \leq N(lat(m)) \tag{29}$$

∗ **Use of symmetry properties of the polynomials of Legendre (global models):** Due to relation (30) scalar fields to be transformed are split into symmetric and antisymmetric parts f_{sym} and f_{ant} matching $f_{\text{sym}}(\Lambda, -\Theta) = f_{\text{sym}}(\Lambda, \Theta)$ and $f_{\text{ant}}(\Lambda, -\Theta) = -f_{\text{ant}}(\Lambda, \Theta)$. Spectral expression of f_{sym} (resp. f_{ant}) include all coefficients with even (resp. odd) value of $m + n$. Legendre transforms are done on f_{sym} and f_{ant} and scalar products defined by equations (20) and (28) can reduce to latitudes $lat1$ to $ndgnh$.

2.9 Relationship between dimension of spectral space and grid point space for Gaussian grid.

∗ **Quadratic grid, linear grid and cubic grids in global models:**

- Spectral space is defined by a triangular truncation N_{s}. Grid point space has $ndglg$ latitudes and a maximum number of longitudes equal to $ndlon$. $ndlon$ and $ndglg$ are always even integers: if $ndlon$ is a multiple of 4, $ndglg = ndlon/2$; if $ndlon$ is not a multiple of 4, $ndglg = ndlon/2 + 1$. The latitudes of the Gaussian grid are computed by an algorithm detailed in appendix 2.

- For a quadratic Gaussian grid, there is a relationship between these parameters to avoid aliasing on quadratic terms. If the stretching coefficient c is equal to 1 (resp. > 1) N_{s} is the biggest integer which matches the relationship $3 * N_{\text{s}} \leq (ndlon - 1)$ (resp. $3 * N_{\text{s}} \leq \min(2 * ndglg - 3, ndlon - 1)$).

- In a semi-Lagrangian scheme the advective quadratic terms disappear, so it is possible to use a smaller grid-point space. The most frequently grid used is the "linear grid": If the stretching coefficient c is equal to 1 (resp. > 1) N_{s} is the biggest integer which matches the relationship $2 * N_{\text{s}} \leq (ndlon - 1)$ (resp. $2 * N_{\text{s}} \leq \min(2 * ndglg - 3, ndlon - 1)$).

- Recently, the concept of cubic grid has been introduced. If the stretching coefficient c is equal to 1 (resp. > 1) N_{s} is the biggest integer which matches the relationship $4 * N_{\text{s}} \leq (ndlon - 1)$ (resp. $4 * N_{\text{s}} \leq \min(2 * ndglg - 3, ndlon - 1)$).

- We generally use a reduced Gaussian grid; the number of grid-points is progressively reduced as we go towards computational poles. In reduced Gaussian grids the total amount of grid-points is around $(2/3)ndlon * ndglg$.

∗ **Quadratic grid, linear grid and cubic grids in LAM models:** similar notions exist in LAM models. For example:
For a quadratic grid:

- N_{s} is the biggest integer which matches the relationship $3 * N_{\text{s}} \leq (ndglg - 1)$.
- N_{ms} is the biggest integer which matches the relationship $3 * N_{\text{ms}} \leq (ndlon - 1)$.

For a linear grid:

- N_{s} is the biggest integer which matches the relationship $2 * N_{\text{s}} \leq (ndglg - 1)$.
- N_{ms} is the biggest integer which matches the relationship $2 * N_{\text{ms}} \leq (ndlon - 1)$.

For a cubic grid:

- N_{s} is the biggest integer which matches the relationship $4 * N_{\text{s}} \leq (ndglg - 1)$.
- N_{ms} is the biggest integer which matches the relationship $4 * N_{\text{ms}} \leq (ndlon - 1)$.

∗ **Admissible dimensions for spherical geometry:** If FFT992 is used, the codes of FFT (fast Fourier transforms) used allows to use integers $ndlon$ which can write as $2^{1+p_2} * 3^{p_3} * 5^{p_5}$. That limits the possibility of choosing the dimensions in a discontinuous subset of truncations and dimensions for Gaussian grid. In the following table one can find the admissible values for $(ndlon, ndglg)$ for $ndglg$ between 32 and 5000, and the corresponding truncations for a quadratic grid and a linear grid. This constraint does not exist with FFTW.

∗ **Admissible dimensions for LAM models:** If FFT992 is used, the codes of FFT (fast Fourier transforms) used allows to use integers $ndlon$ and $ndglg$ which can write as $2^{1+p_2} * 3^{p_3} * 5^{p_5}$. This constraint does not exist with FFTW.

$(ndglg, ndlon)$	Cub	Quad $c>1$	Quad $c=1$	Lin $c>1$	Lin $c=1$	$(ndglg, ndlon)$	Cub	Quad $c>1$	Quad $c=1$	Lin $c>1$	Lin $c=1$
(32, 64)	15	20	21	30	31	(864,1728)	431	575	575	862	863
(36, 72)	17	23	23	34	35	(900,1800)	449	599	599	898	899
(40, 80)	19	25	26	38	39	(960,1920)	479	639	639	958	959
(46, 90)	22	29	29	44	44	(972,1944)	485	647	647	970	971
(48, 96)	23	31	31	46	47	(1000,2000)	499	665	666	998	999
(50, 100)	24	32	33	48	49	(1024,2048)	511	681	682	1022	1023
(54, 108)	26	35	35	52	53	(1080,2160)	539	719	719	1078	1079
(60, 120)	29	39	39	58	59	(1126,2250)	562	749	749	1124	1124
(64, 128)	31	41	42	62	63	(1152,2304)	575	767	767	1150	1151
(72, 144)	35	47	47	70	71	(1200,2400)	599	799	799	1198	1199
(76, 150)	37	49	49	74	74	(1216,2430)	607	809	809	1214	1214
(80, 160)	39	52	53	78	79	(1250,2500)	624	832	833	1248	1249
(82, 162)	40	53	53	80	80	(1280,2560)	639	852	853	1278	1279
(90, 180)	44	59	59	88	89	(1296,2592)	647	863	863	1294	1295
(96, 192)	47	63	63	94	95	(1350,2700)	674	899	899	1348	1349
(100, 200)	49	65	66	98	99	(1440,2880)	719	959	959	1438	1439
(108, 216)	53	71	71	106	107	(1458,2916)	728	971	971	1456	1457
(120, 240)	59	79	79	118	119	(1500,3000)	749	999	999	1498	1499
(126, 250)	62	83	83	124	124	(1536,3072)	767	1023	1023	1534	1535
(128, 256)	63	84	85	126	127	(1600,3200)	799	1065	1066	1598	1599
(136, 270)	67	89	89	134	134	(1620,3240)	809	1079	1079	1618	1619
(144, 288)	71	95	95	142	143	(1728,3456)	863	1151	1151	1726	1727
(150, 300)	74	99	99	148	149	(1800,3600)	899	1199	1199	1798	1799
(160, 320)	79	105	106	158	159	(1876,3750)	937	1249	1249	1874	1874
(162, 324)	80	107	107	160	161	(1920,3840)	959	1279	1279	1918	1919
(180, 360)	89	119	119	178	179	(1944,3888)	971	1295	1295	1942	1943
(192, 384)	95	127	127	190	191	(2000,4000)	999	1332	1333	1998	1999
(200, 400)	99	132	133	198	199	(2026,4050)	1012	1349	1349	2024	2024
(216, 432)	107	143	143	214	215	(2048,4096)	1023	1364	1365	2046	2047
(226, 450)	112	149	149	224	224	(2160,4320)	1079	1439	1439	2158	2159
(240, 480)	119	159	159	238	239	(2188,4374)	1093	1457	1457	2186	2186
(244, 486)	121	161	161	242	242	(2250,4500)	1124	1499	1499	2248	2249
(250, 500)	124	165	166	248	249	(2304,4608)	1151	1535	1535	2302	2303
(256, 512)	127	169	170	254	255	(2400,4800)	1199	1599	1599	2398	2399
(270, 540)	134	179	179	268	269	(2430,4860)	1214	1619	1619	2428	2429
(288, 576)	143	191	191	286	287	(2500,5000)	1249	1665	1666	2498	2499
(300, 600)	149	199	199	298	299	(2560,5120)	1279	1705	1706	2558	2559
(320, 640)	159	212	213	318	319	(2592,5184)	1295	1727	1727	2590	2591
(324, 648)	161	215	215	322	323	(2700,5400)	1349	1799	1799	2698	2699
(360, 720)	179	239	239	358	359	(2880,5760)	1439	1919	1919	2878	2879
(376, 750)	187	249	249	374	374	(2916,5832)	1457	1943	1943	2914	2915
(384, 768)	191	255	255	382	383	(3000,6000)	1499	1999	1999	2998	2999
(400, 800)	199	265	266	398	399	(3072,6144)	1535	2047	2047	3070	3071
(406, 810)	202	269	269	404	404	(3126,6250)	1562	2083	2083	3124	3124
(432, 864)	215	287	287	430	431	(3200,6400)	1599	2132	2133	3198	3199
(450, 900)	224	299	299	448	449	(3240,6480)	1619	2159	2159	3238	3239
(480, 960)	239	319	319	478	479	(3376,6750)	1687	2249	2249	3374	3374
(486, 972)	242	323	323	484	485	(3456,6912)	1727	2303	2303	3454	3455
(500,1000)	249	332	333	498	499	(3600,7200)	1799	2399	2399	3598	3599
(512,1024)	255	340	341	510	511	(3646,7290)	1822	2429	2429	3644	3644
(540,1080)	269	359	359	538	539	(3750,7500)	1874	2499	2499	3748	3749
(576,1152)	287	383	383	574	575	(3840,7680)	1919	2559	2559	3838	3839
(600,1200)	299	399	399	598	599	(3888,7776)	1943	2591	2591	3886	3887
(626,1250)	312	416	416	624	624	(4000,8000)	1999	2665	2666	3998	3999
(640,1280)	319	425	426	638	639	(4050,8100)	2024	2699	2699	4048	4049
(648,1296)	323	431	431	646	647	(4096,8192)	2047	2729	2730	4094	4095
(676,1350)	337	449	449	674	674	(4320,8640)	2159	2879	2879	4318	4319
(720,1440)	359	479	479	718	719	(4374,8748)	2186	2915	2915	4372	4373
(730,1458)	364	485	485	728	728	(4500,9000)	2249	2999	2999	4498	4499
(750,1500)	374	499	499	748	749	(4608,9216)	2303	3071	3071	4606	4607
(768,1536)	383	511	511	766	767	(4800,9600)	2399	3199	3199	4798	4799
(800,1600)	399	532	533	798	799	(4860,9720)	2429	3239	3239	4858	4859
(810,1620)	404	539	539	808	809	(5000,10000)	2499	3332	3333	4998	4999

3   Set-up routines.

∗ Organigramme for spherical geometry:

```
SUOYOMA ->
- SUTRANS0 ->
  * SETUP_TRANS0 -> SUMP_TRANS0 ->
      - PE2SET
      - MPL_GROUPS_CREATE
- SUGEOMETRY -> SUTRANS ->
  * SETUP_TRANS ->
      - SET_RESOL
      - SETUP_DIMS
      - SUMP_TRANS_PRELEG -> SUWAVEDI
      - SULEG ->
          * INI_POL
          * SUGAW ->
            - GAWL -> CPLEDN
            - SUPOLF
          * SUPOL
          * SUTRLE -> SET2PE and routines MPL_...
          * SETUP_GEOM
          * SUPOLF (for fast Legendre transforms only?)
          * CONSTRUCT_BUTTERFLY (for fast Legendre transforms only)
      - SUMP_TRANS ->
          * SUMPLATF -> SUMPLATB or SUMPLATBEQ
          * SUMPLAT -> SUMPLATB or SUMPLATBEQ
          * SUSTAONL -> SET2PE and MPL_... routines
      - SUFFT ->
          * SET99 (use of FFT992)
          * INIT_PLANS_FFTW (use of FFTW)
- SULEGA
```

∗ Organigramme for LAM models:

```
SUOYOMA ->
- SUETRANS0 ->
  * SETUP_TRANS0 -> SUMP_TRANS0 ->
      - PE2SET
      - MPL_GROUPS_CREATE
- SUGEOMETRY -> SUETRANS ->
  * ESETUP_TRANS ->
      - ESET_RESOL
      - ESETUP_DIMS
      - SUEMP_TRANS_PRELEG
      - ESETUP_GEOM
      - SUEMP_TRANS ->
          * SUMPLATF -> SUMPLATB
          * SUEMPLAT -> SUEMPLATB
          * SUESTAONL -> SET2PE and MPL_... routines
      - SUEFFT -> SET99
- SUELEGA
```

∗ Computations done by these routines:

- For Legendre transforms, one needs to compute polynomials of Legendre and some other quantities, like Gaussian latitudes, Gaussian weights, geometrical quantities containing $\cos\Theta$. Such computations are done in routine **SULEG** and subroutines called by **SULEG**.

- For "FFT992" Fourier transforms, one needs to compute quantities **TRIGS** and **NFAX** necessary to use routine **FFT992**. These two arrays are computed in routines **SUFFT** and **SUEFFT** (note that a call to **FFT992** reinitialises them).

- For "FFTW" Fourier transforms, the set-up is done in **INIT_PLANS_FFTW**.

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- **SUGAW** computes a first estimation of Gaussian latitudes, then calls **GAWL**. It is required in spherical geometry only. There is one call of **GAWL** for each latitude. **GAWL** manages an iterative algorithm for one latitude: one call to **CPLEDN** corresponds to one iteration. Computations stop once the iterative algorithm converges towards a satisfactory solution. So one obtains the Gaussian latitude and weight. Details of the algorithm implemented in **SUGAW** and called routines is described in appendix 2.

- Exiting **SUGAW** the sine of Gaussian latitude $\mu = \sin \Theta$ is available in real array **YRCSGLEG%RMU**. Gaussian weights are stored in global array **YRCSGLEG%RW**. The following attributes of **YRCSGLEG** are then computed in routine **SULEGA**:

  - **R1MU2**: $\cos^2 \Theta = 1 - \mu^2$
  - **R1MUI**: $\frac{1}{\cos^2 \Theta} = \frac{1}{1-\mu^2}$
  - **R1MUA**: $\frac{1}{a \cos^2 \Theta} = \frac{1}{a(1-\mu^2)}$
  - **RSQM2**: $\cos \Theta = \sqrt{1 - \mu^2}$
  - **R1QM2**: $\frac{1}{\cos \Theta} = \frac{1}{\sqrt{1-\mu^2}}$
  - **RACTHE**: $\frac{1}{a \cos \Theta} = \frac{1}{a\sqrt{1-\mu^2}}$
  - **RLATIG**: Θ

- **SUPOL** computes the polynomials of Legendre. One call of **SUPOL** performs computations for one latitude only. Computations are done in 128 bits precision, following an algorithm described in appendix 3.

- **SUPOLF** is an alternate version of **SUPOL** (for fast Legendre transforms?).

- Exiting **SUPOL** yields the array **RPNM** containing the polynomials of Legendre. For each Gaussian latitude, **RPNM** is an array with a structure identical to spectral arrays. **RPNM** are stored only for the Northern Hemisphere (of computational sphere) due to the following property:

$$P_{(n,m)}(-\mu) = (-1)^{(m+n)} P_{(n,m)}(\mu) \tag{30}$$

- Some other quantities involved in the spectral transforms are computed in (or under) the following routines: **SETUP_DIMS**, **SETUP_GEOM_MOD**, **SET_RESOL**, **SUMP_TRANS_PRELEG**, **SETUP_GEOM**, **SUMP_TRANS** and their LAM counterparts (names starting by **ESETUP**, **SUEMP**).
  For more details, see comments inside these routines.

$*$ **Distributed memory:** Routine **SUTRLE** manages transpositions in the distributed memory environment. For distributed memory, computations of the Legendre polynomials are DM-local for Gaussian latitudes (a processor does computations only in a subset of latitudes) and DM-global for spectral coefficients (a processor computes all the spectral coefficients for this subset of latitudes). So one first computes an intermediate Legendre polynomials array with DM-local indexation for Gaussian latitudes and DM-global indexation for spectral coefficients. Then a memory transfer with communications between processors is necessary to obtain the final Legendre polynomials **RPNM** which has a DM-global indexation for Gaussian latitudes (all latitudes are available in the same processor) and a DM-local indexation for spectral coefficients (contains only a subset of zonal wave numbers). **SET2PE** gives the number of the processor between 1 and **NPROC**, once known the following numbers:

- the "A-set" number of the processor between 1 and **NPRGPNS** for grid-point computations.
- the "B-set" number of the processor between 1 and **NPRGPEW** for grid-point computations.
- the "A-set" number of the processor between 1 and **NPRTRW** for spectral computations.
- the "B-set" number of the processor between 1 and **NPRTRV** for spectral computations.

PE2SET does the reverse thing of **SET2PE**. The **MPL_...** routines are message-passing routines doing communication between processors. For more details about them see documentation (IDDM).

$*$ **Duplicate version for spherical geometry:** Most of the variables computed under **SUTRANS** have a "duplicate version" in the "arpifs" library, sometimes computed separately in "arpifs", because they are used in other parts of the ARPEGE code. The "trans" version (in modules **TPM_...**) is used in the spectral transforms, the "arpifs" version is used elsewhere.

$*$ **Duplicate version for LAM models:** Most of the variables computed under **SUETRANS** have a "duplicate version" in the "arpifs" library, sometimes computed separately in "arpifs", because they are used in other parts of the ARPEGE/ALADIN code. The "etrans" version (in modules **TPMALD_...**) is used in the spectral transforms, the "arpifs" version is used elsewhere.

4 Spectral transforms routines: the general routines (E)DIR_TRANS and (E)INV_TRANS.

4.1 Range of use.

They can be used for a set of fields (model fields), but they use optional arguments, so they can also be used for only one separate field. **DIR_TRANS** (**EDIR_TRANS** for LAM models) does direct transforms (from grid-point space to spectral space), **INV_TRANS** (**EINV_TRANS** for LAM models) does inverse transforms (from spectral space to grid-point space). These routines have adjoint versions **(E)DIR_TRANSAD** and **(E)INV_TRANSAD**. The tangent linear code is identical to the direct code. These routines and the routines called under them are in the "trans" library for spherical geometry, "etrans" library for LAM models.

4.2 Organigrammes for direct code.

∗ **Organigrammes of INV_TRANS and EINV_TRANS:**

```
INV_TRANS ->                            | EINV_TRANS ->
  * SET_RESOL                           |   * ESET_RESOL
  * INV_TRANS_CTL ->                    |   * EINV_TRANS_CTL ->
    - SHUFFLE (case NPRTRV>1)           |     - SHUFFLE (case NPRTRV>1)
    - FIELD_SPLIT                       |     - FIELD_SPLIT
    - LTINV_CTL ->                      |     - ELTINV_CTL ->
      * LTINV ->                        |       * ELTINV ->
        - PREPSNM                       |
        - PRFI1B                        |         - EPRFI1B
        - VDTUV                         |         - EVDTUV
        - SPNSDE                        |         - ESPNSDE
        - LEINV -> DGEMM                |         - ELEINV -> FFT992 or EXEC_EFFTW
                  MULT_BUTM (FLT)       |
        - ASRE1B                        |         - EASRE1B
        - FSPGL_INT                     |         - FSPGL_INT
      * TRMTOL -> (MPL routines)        |       * TRMTOL -> (MPL routines)
    - FTINV_CTL ->                      |     - EFTINV_CTL ->
      * FOURIER_IN                      |       * FOURIER_IN
      * FSC                             |       * EFSC
      * FTINV ->                        |       * FTINV ->
        - FFT992 or EXEC_FFTW           |         - FFT992 or EXEC_EFFTW
      * TRLTOG -> (MPL routines)        |       * TRLTOG -> (MPL routines)
```

∗ **Organigrammes of DIR_TRANS and EDIR_TRANS:**

```
DIR_TRANS ->                            | EDIR_TRANS ->
  * SET_RESOL                           |   * ESET_RESOL
  * DIR_TRANS_CTL ->                    |   * EDIR_TRANS_CTL ->
    - SHUFFLE (case NPRTRV>1)           |     - SHUFFLE (case NPRTRV>1)
    - FIELD_SPLIT                       |     - FIELD_SPLIT
    - FTDIR_CTL ->                      |     - EFTDIR_CTL ->
      * TRGTOL -> (MPL routines)        |       * TRGTOL -> (MPL routines)
                                        |       * EXTPER
                                        |       * AUX_PROC
      * FTDIR ->                        |       * FTDIR ->
        - FFT992 or EXEC_FFTW           |         - FFT992 or EXEC_EFFTW
      * FOURIER_OUT                     |       * FOURIER_OUT
    - LTDIR_CTL ->                      |     - ELTDIR_CTL ->
      * TRLTOM -> (MPL routines)        |       * TRLTOM -> (MPL routines)
                                        |       * AUX_PROC
      * LTDIR ->                        |       * ELTDIR ->
        - PRFI2 -> PRFI2B               |         - EPRFI2 -> EPRFI2B
                                        |         - EXTPER
        - LDFOU2                        |
        - LEDIR -> DGEMM                |         - ELEDIR -> FFT992 or EXEC_EFFTW
                  MULT_BUTM (FLT)       |
        - PREPSNM                       |
        - UVTVD                         |         - EUVTVD
                                        |         - EUVTVD_COMM
        - UPDSP -> UPDSPB               |         - EUPDSP -> EUPDSPB
```

∗ **Work done by routines called under INV_TRANS_CTL and DIR_TRANS_CTL:**

- **LTINV_CTL**: upper control routine for inverse Legendre transforms.

- **LTDIR_CTL**: upper control routine for direct Legendre transforms.
- **LTINV**: lower control routine for inverse Legendre transforms.
- **LTDIR**: lower control routine for direct Legendre transforms.
- **FTINV_CTL**: upper control routine for inverse Fourier transforms.
- **FTDIR_CTL**: upper control routine for direct Fourier transforms.
- **FTINV**: lower control routine for inverse Fourier transforms.
- **FTDIR**: lower control routine for direct Fourier transforms.
- **PREPSNM**: stores useful values of $e_{(n,m)}$ (in array **REPSNM**) for a given zonal wave number m.
- **PRFI1B**: stores useful spectral coefficients of the fields to be transformed for a given zonal wave number m and latitudes $lat1(m)$ to $ndgnh$ in a work array **PIA**.
- **PRFI2B**: splits Fourier coefficients into symmetrical and antisymmetrical parts; stores useful Fourier coefficients of the fields to be transformed for a given zonal wave number m and latitudes $lat1(m)$ to $ndgnh$ in a work array **PSIA** for the symmetrical part and **PAIA** for the antisymmetrical part.
- **PRFI2**: interface for **PRFI2B** (one call to **PRFI2B** for scalars, two for vectors).
- **VDTUV**: used only when transform of vector wind is required. Computation of $(a\cos\Theta U', a\cos\Theta V')$ in spectral space knowing (D', ζ').
- **UVTVD**: computation of (D', ζ') in spectral space knowing $\left(\frac{U'}{a\cos\Theta}, \frac{V'}{a\cos\Theta}\right)$.
- **SPNSDE**: computation of meridian derivatives when required.
- **LDFOU2**: division by $a\cos\Theta$ in Fourier space for wind.
- **LEINV** performs two inverse Legendre transforms, the first one for f_{sym}, the second one for f_{ant}, that gives the Fourier coefficients stored respectively in arrays **PSOA1** and **PAOA1** for symmetrical and antisymmetrical parts.
- **LEDIR** performs two direct Legendre transforms, the first one for $F_{\text{sym }m}(\mu)$, the second one for $F_{\text{ant }m}(\mu)$, that gives the spectral coefficients stored in the array **POA1**.
- **DGEMM**: product matrix by matrix.
- **MULT_BUTM**: product matrix by matrix with the butterfly method.
- **ASRE1B** recombines symmetrical and antisymmetrical parts to compute Fourier coefficients $F_m(\mu)$.

$$F_m(\mu) = F_{\text{sym }m}(\mu) + F_{\text{ant }m}(\mu) \ ; \quad F_m(-\mu) = F_{\text{sym }m}(\mu) - F_{\text{ant }m}(\mu)$$

  Exiting **ASRE1B** Fourier coefficients are in the array **PREEL**. For a latitude index jgl the Fourier space truncation is $N(jgl)$ available in the array **NMEN**. Values are set to zero in the array **PREEL** for zonal waves numbers m greater than $N(jgl)$.
- **UPDSPB**: final memory transfer in spectral array containing all coefficients.
- **UPDSP**: interface routine for **UPDSPB**.
- **FOURIER_IN**: separates the real and imaginary parts of the Fourier coefficients.
- **FOURIER_OUT**: recombines the real and imaginary parts of the Fourier coefficients.
- **FSC**: division by $a\cos\Theta$ in Fourier space for wind, and first order derivatives.
- **FFT992** performs Fourier transforms for "FFT992" package.
- **EXEC_FFTW** is called instead of **FFT992** if "FFTW" package is used.
- **TRLTOM**: transposition routine for distributed memory. Transpose Fourier buffer data from partitioning over latitudes (necessary for direct Fourier transforms) to partitioning over wave numbers (necessary for direct Legendre transforms).
- **TRMTOL**: does the inverse transposition of **TRLTOM**.
- **TRGTOL**: transposition routine for distributed memory. Transpose grid-point data from partitioning over sets of **NGPTOT** grid-points containing complete columns (all the **NFLEVG** model layers) and incomplete latitudes to partitioning over complete latitudes (a processor contains a subset of complete latitudes) and layers if **NPRTRV**>1 (a processor contains **NFLEVL** layers).
- **TRLTOG**: does the inverse transposition of **TRGTOL**.
- **SHUFFLE**: re-shuffle fields for load balancing if **NPRTRV**>1. Note that the relative order of the local spectral fields has to maintained.
- **FIELD_SPLIT**: split fields for case **NPRTRV**>1; simple memory transfer if **NPRTRV**=1.
- **FSPGL_INT**: interface for calculations done in Fourier space.

∗ Work done by routines called under EINV_TRANS_CTL and EDIR_TRANS_CTL:

We can notice that we generally have the same routines as for "trans" routines, with an additional "E" at the beginning of their names when a specific "etrans" routine is required: for example **ELEDIR** instead of **LEDIR**. Note the following specificies:
- **ELEDIR** and **ELEINV** do meridian Fourier transforms and call **FFT992**.
- **AUX_PROC** is called to do periodicisation of auxiliary fields.
- **EXTPER** is called to do periodicisation when there is no extension zone.

4.3 Organigrammes for adjoint code.

Organigramme is given for **INV_TRANSAD** and **DIR_TRANSAD**; their LAM counterparts **EINV_TRANSAD** and **EDIR_TRANSAD** also exist (organigramme not detailed).

∗ **Organigramme of INV_TRANSAD:**

```
INV_TRANSAD ->
  * SET_RESOL
  * INV_TRANS_CTLAD ->
    - SHUFFLE (case NPRTRV>1)
    - FIELD_SPLIT
    - FTINV_CTLAD ->
      * TRGTOL -> (MPL routines)
      * FTINVAD ->
        - FFT992 or EXEC_FFTW
      * FSCAD
      * FOURIER_INAD
    - LTINV_CTLAD ->
      * TRLTOM -> (MPL routines)
      * LTINVAD ->
        - PREPSNM
        - ASRE1BAD
        - LEINVAD -> DGEMM
                    MULT_BUTM (FLT)
        - VDTUVAD
        - SPNSDEAD
        - PRFI1BAD
```

∗ **Organigramme of DIR_TRANSAD:**

```
DIR_TRANSAD ->
  * SET_RESOL
  * DIR_TRANS_CTLAD ->
    - SHUFFLE (case NPRTRV>1)
    - FIELD_SPLIT
    - LTDIR_CTLAD ->
      * LTDIRAD ->
        - UPDSPAD -> UPDSPBAD
        - PREPSNM
        - UVTVDAD
        - LEDIRAD -> DGEMM
                    MULT_BUTM (FLT)
        - LDFOU2AD
        - PRFI2AD -> PRFI2BAD
      * TRMTOL -> (MPL routines)
    - FTDIR_CTLAD ->
      * FOURIER_OUTAD
      * FTDIRAD ->
        - FFT992 or EXEC_FFTW
      * TRLTOG -> (MPL routines)
```

5 Spectral transforms routines for general application.

5.1 Spectral transforms routines for scalar variables: SPEREE and REESPE.

REESPE does direct transforms (from grid-point space to spectral space), **SPEREE** does inverse transforms (from spectral space to grid-point space). **SPEREE** calls **INV_TRANS**; **REESPE** calls **DIR_TRANS**; the adjoint **SPERAD** of **SPEREE** calls **REESPE**. These routines perform transforms for scalar fields according to the algorithm described in subsection (2.8).

These routines have LAM counterparts: **EREESPE** and **ESPEREE** (direct code and TL code), **ESPERAD** (adjoint code).

5.2 Wind components: routines SPEUV and UVSPE.

∗ **Transformation from spectral to grid point space.** D' is the reduced divergence, ζ' is the reduced vorticity. χ is the velocity potential, ψ is the stream function. U' and V' are the reduced wind components. D' and ζ' are available in spectral space. Transformation from spectral to grid point space needs:

- A transformation $(\chi, \psi) \rightarrow (D', \zeta')$ if required.
- Computation of $(a \cos \Theta U', a \cos \Theta V')$ in spectral space knowing (D', ζ'), following formulae (14) and (15).
- Inverse Legendre transform on $a \cos \Theta U'$ and $a \cos \Theta V'$.
- In Fourier space, division by $a \cos \Theta$ for each latitude: yields U' and V' in this space.
- Inverse Fourier transform on U' and V' for each latitude.

∗ **Transformation from grid point to spectral space.** Transformation from grid point to spectral space needs:

- Direct Fourier transform on U' and V' for each latitude.
- Computation of $\left(\frac{U'}{a \cos \Theta}, \frac{V'}{a \cos \Theta} \right)$ in Fourier space for each latitude.
- Direct Legendre transform on $\frac{U'}{a \cos \Theta}$ and $\frac{V'}{a \cos \Theta}$. Spectral coefficients obtained after this transform correspond to the $\tilde{U}_{(n,m)}$ and $\tilde{V}_{(n,m)}$ defined in equations (2.24) and (2.25) of (Temperton, 1991).
- Computation of (D', ζ') in spectral space knowing $\left(\frac{U'}{a \cos \Theta}, \frac{V'}{a \cos \Theta} \right)$, following formulae (2.26) and (2.27) of (Temperton, 1991).
- A transformation $(D', \zeta') \rightarrow (\chi, \psi)$ if required.

∗ **Routines performing transformations:** **SPEUV** performs a transformation from spectral space (input variables are (D', ζ') or (χ, ψ)) to grid-point space (output variables are U' and V'). **UVSPE** performs a transformation from grid-point space (input variables are U' and V') to spectral space (output variables are (D', ζ') if **KOUTP**=1 or (χ, ψ) if **KOUTP**=2). **SPEUV** calls **INV_TRANS**; **UVSPE** calls **DIR_TRANS**.

∗ **LAM model versions:**

- **ESPEUV** is the LAM counterpart of **SPEUV**.
- There are also **EUVGEOVD** (retrieve divergence and vorticity from wind components in spectral space) and **EVDUVGEO** (retrieve wind components from divergence and vorticity in spectral space).

6 Spectral transforms used in the model under routine STEPO and for specific applications.

6.1 Organigramme under STEPO.

In model integration, routines **TRANSINVH** and **TRANSDIRH** play the same part as **SPEREE**/**SPEUV**, **REESPE**/**UVSPE**, but they deal with a pre-determined set of variables. Horizontal derivatives of these variables have generally to be computed during the inverse transform. Meridian derivatives are computed after converting $(D^{'}, \zeta^{'})$ into $(a \cos \Theta U^{'}, a \cos \Theta V^{'})$ and before inverse Legendre transforms in spectral space (call to routine **SPNSDE**). These spectral transforms routines are also used for the spectral fit of FULL-POS.

∗ **General organigramme under STEPO:** one starts from routine **STEPO** which controls one time-step.

- **control/STEPO** →
    - Data reading/writing
    - **(E)TRANSINVH** (see below the detailed organigramme)
    - **SCAN2M** (grid point computations)
    - **OBSV** (for assimilation part computing J_{o})
    - Diagnostics
    - **(E)TRANSDIRH** (see below the detailed organigramme)
    - Spectral computations

FULLPOS has its own version of STEPO (**STEPO_FPOS**) calling TRANSINV_FPOS and TRANSDIR_FPOS.

∗ **Organigramme under (E)TRANSINVH:**

```
(E)TRANSINVH -> (E)TRANSINV_MDL -> (E)INV_TRANS (model variables)
```

∗ **Organigramme under (E)TRANSDIRH:**

```
(E)TRANSDIRH -> (E)TRANSDIR_MDL -> (E)DIR_TRANS (model variables)
```

∗ **Tangent linear code:** The tangent linear code of **(E)TRANSINVH** (called under **STEPOTL**) is **(E)TRANSINVH** itself. The tangent linear code of **(E)TRANSDIRH** (called under **STEPOTL**) is **(E)TRANSDIRH** itself.

∗ **Adjoint code:** The adjoint code of **(E)TRANSINVH** (called under **STEPOAD**) is **(E)TRANSINVHAD**. The adjoint code of **(E)TRANSDIRH** (called under **STEPOAD**) is **(E)TRANSDIRHAD**. Organigrammes are:

```
(E)TRANSINVHAD -> (E)TRANSINV_MDLAD -> (E)INV_TRANSAD
```

and:

```
(E)TRANSDIRHAD -> (E)TRANSDIR_MDLAD -> (E)DIR_TRANSAD
```

6.2 Spectral transforms necessary for model integration.

Only model integration is here presented. For other applications where **TRANSDIRH** or **TRANSINVH** are used (assimilation, initialisation by DFI), see corresponding documentations. Variables are divided in three classes:

- GMV (3D) variables: the adiabatic part of the RHS of their equation is non-zero and these variables have always a spectral representation (example: wind components, temperature).
- GMVS (2D) variables: the adiabatic part of the RHS of their equation is non-zero and these variables have always a spectral representation (example: logarithm of hydrostatic surface pressure).
- GFL variables: the adiabatic part of the RHS of their equation is zero and these variables can have a spectral representation or not (example: specific humidity, ozone).

For more details about this classification, see documentation (IDEUL).

6.2.1 3D primitive equation hydrostatic model.

∗ **Input variables in spectral space to be transformed into grid-point space:** For GMV variables: reduced divergence $D^{'}$, reduced vorticity $\zeta^{'}$, temperature T if **LSPRT**=.F. or RT if **LSPRT**=.T.. For GMVS variables: pressure surface variable $\log \Pi_{\mathrm{s}}$ (Π_{s} is the surface pressure). All the GFL variables which have a spectral representation.

∗ **Output variables to be obtained in grid-point space:** Variable X, meridian derivative $\frac{1}{a\cos\Theta}\frac{\partial X}{\partial\Theta}$ and zonal derivative $\frac{1}{a}\frac{\partial X}{\partial\Lambda}$ for the following variables: reduced divergence $D^{'}$, reduced vorticity $\zeta^{'}$, reduced wind components $U^{'}$ and $V^{'}$, temperature T if **LSPRT**=.F. or RT if **LSPRT**=.T., pressure surface variable $\log \Pi_{\mathrm{s}}$, all the GFL variables which have a spectral representation. All horizontal derivatives are required in an Eulerian model but some derivatives are useless in a semi-Lagrangian model.

∗ **Input variables in grid-point space to be transformed into spectral space:** Provisional $t + \Delta t$ quantities before semi-implicit scheme and horizontal diffusion scheme for GMV variables equations (momentum equation, temperature equation (virtual temperature equation if **LSPRT**=.T.)), GMVS variables equations (continuity equation), GFL equations for GFL variables which have a spectral representation. For momentum equation input variables are reduced wind components, output variables are reduced divergence and vorticity.

6.2.2 3D non-hydrostatic model.

∗ **Input/Output variables:** See part 6.2.1 for the list of variables which exist in the hydrostatic model. Two additional variables (one linked to the pressure departure, the second one linked to the vertical divergence) must be transformed with the same treatment of temperature. In some cases a quantity denoted "NHX" (X-term) must be also transformed.

∗ **Additional spectral transforms necessary for iterative centred-implicit schemes:** The iterative part of these schemes do spectral transforms as for the predictive part (transforms are done on the provisional $t + \Delta t$ quantities obtained after each iteration); transforms involves all variables or a subset of them according to the type of centred-implicit scheme used.

6.2.3 Shallow-water 2D model.

∗ **Input variables in spectral space to be transformed into grid-point space:** Reduced divergence $D^{'}$, reduced vorticity $\zeta^{'}$, equivalent height Φ.

∗ **Output variables to be obtained in grid-point space:** Variable X, meridian derivative $\frac{1}{a\cos\Theta}\frac{\partial X}{\partial\Theta}$ and zonal derivative $\frac{1}{a}\frac{\partial X}{\partial\Lambda}$ for the following variables: reduced divergence $D^{'}$, reduced vorticity $\zeta^{'}$, reduced wind components $U^{'}$ and $V^{'}$, equivalent height Φ. All horizontal derivatives are required in an Eulerian model but some derivatives are useless in a semi-Lagrangian model.

∗ **Input variables in grid-point space to be transformed into spectral space:** Provisional $t + \Delta t$ quantities before semi-implicit scheme and horizontal diffusion scheme for momentum equation and continuity equation. For momentum equation input variables are reduced wind components, output variables are reduced divergence and vorticity.

6.3 Spectral transforms necessary for particular applications.

One finds some other **TRANSDIR_...** (which call **DIR_TRANS**) and **TRANSINV_....** (which call **INV_TRANS**) routines (resp. **ETRANSDIR_....** and **ETRANSINV_....** routines for LAM models) which are designed for specific applications. For example, for ARPEGE, one finds:

- **TRANSDIR_NHCONV** and **TRANSINV_NHCONV**: interface routine for the spectral transforms which are required when converting the data read on files towards the prognostic variables (and the reverse one) in the non-hydrostatic model.

- **TRANSDIR_NHCONVPRHS** and **TRANSINV_NHCONVPRHS**: same type of application as the previous one; additional transforms required for Wood and Staniforth formulation of deep-layer NH equations.

- **TRANSDIR_WAVELET** and **TRANSINV_WAVELET**: interface routine for the spectral transforms which are required to compute the wavelet J_{b} in a 3D-VAR or 4D-VAR assimilation using a wavelet representation. These routines have adjoints.

7 Treatment of transmission coefficients when storing them in Fourier space.

∗ **Introduction:** The structure of the code does not allow to have a distributed memory code which gives **NPROMA**-dimensioned grid-point arrays for transmission coefficients without the intermediate stage of **NGPTOT**-dimensioned grid-point arrays. The problem comes principally from Fourier transforms and not from Legendre transforms. So a new code structure has been developed:

- Fourier coefficients instead of spectral coefficients are stored on files. No Legendre transform is done.
- Files are currently LFI files (code with ARPEGE files to be provided later).
- Fourier transforms are done by special routines called under **MF_PHYS** on **NPROMA**-dimensioned grid-point arrays, so a DM-code can be provided without a big increase of memory, at least when reading coefficients.
- The "trans" library is not used.

∗ **Setup and data reading:**

- Writing Fourier coefficients is controlled by **LWRTCFOU** in **NAMCT0**. In this case the Fourier coefficients written are a temporal average from the timesteps number 2 to **NSTOP**. **NSTOP**=2 provides instantaneous coefficients.
- Reading Fourier coefficients is controlled by **LRETCFOU** in **NAMCT0**.
- Reading Fourier coefficients on files is done by the following sequence:
  SUSPEC → **SUSPECTCFOU**.
  SUSPECTCFOU reads data on files and calls some diagnostics (routine **FOUTCNORM**). For DM code these actions are done on the "main" processor and data are then sent to the other processors by routine **DISFOU**. For the Curtis matrix, the matrix read on file can be not horizontally constant, but **SUSPECTCFOU** replaces it by its horizontal average because the simplified physics can currently use a horizontally constant quantity for this variable.
- Fourier transforms need to compute a basis of sines/cosines put in (**NTCMAX**,**NGPTOT**)-dimensioned DM-local grid-point arrays (like arrays describing geometry). This is done by the following sequence: **SU0YOMB** → **SURCOFTC** → **SURCOF**. **SURCOFTC** does some allocations, **SURCOF** does calculations. Basis of sines (i.e. $\sin(m\Lambda(jlon))$) is stored in array **RBASSITC**, basis of cosines (i.e. $\cos(m\Lambda(jlon))$) is stored in array **RBASCOTC**. The array **TRIGS** computed in **SUFFT** (project "trans") has not a proper dimension to be used in this case.

∗ **Fourier transforms:**

- Inverse transforms are done in **CPPFTTCINV** called under **MF_PHYS**. Code allows that latitudes can be split between processors or **NPROMA**-packets.
- Direct transforms are done in **CPPFTTCDIR** called under **MF_PHYS**. Currently the code does not allow that latitudes can be split between processors or **NPROMA**-packets. For the DM-code that means that the only admissible value of **NPROMA** is **NGPTOTMX**.
- These two routines call grid-point diagnostics (**GPTCNORM**).

∗ **Data writing:** Writing Fourier coefficients on files is done by the following sequence:
IOPACK → **WRTCFOU**.
WRTCFOU writes data on files and call some diagnostics (routine **FOUTCNORM**). For DM code these actions are done on the "main" processor after that data have been collected from the other processors by routine **DIWRFOU**.

∗ **Adjoint and linear tangent codes:** **CPPFTTCINV** and **CPPFTTCDIR** are also called under **MF_PHYS_AD** and **MF_PHYS_AD** for configurations which use adjoint and tangent linear codes of the model. There are no adjoint routines coded nor used for **CPPFTTCINV** and **CPPFTTCDIR**.

8 Some distributed memory features.

8.1 Case LEQ_REGIONS=F.

∗ **Grid-point computations:** The total number of processors involved in the A-level parallelisation is **NPRGPNS**. The total number of processors involved in the B-level parallelisation is **NPRGPEW**. One processor treats **NGPTOT** points (a part of the Gaussian grid points, latitudes are not always complete). The total amount of grid-points for all the processors is **NGPTOTG**. The maximum value of **NGPTOT** is **NGPTOTMX**. In the grid-point space there is a subdivision of the current processor grid-points into **NGPBLKS**=int[(**NGPTOT**+**NPROMA**-1)/**NPROMA**] packets of length **NPROMA** (the useful number of values in each packet is lower or equal than **NPROMA**). These packets do not contain neither extra-longitudes nor polar or extra-polar latitudes data. A **NPROMA**-packet does not always contain a set of complete latitudes.

One 2D field has **NGPTOTG** points divided into **NPROC** sets of **NGPTOT** points treated by each processor. **NGPTOT** does not take account of the extra-longitudes, the poles and the extra-polar latitudes. All these variables take account of the reduced Gaussian grid. It is assumed and hardcoded that there are one western extra-longitude and two eastern extra-longitudes. The DM-global longitude $jlon = 1$ is always the "Greenwich" meridian of the computational sphere. All the vertical levels and the variables corresponding to a same grid-point are treated by the same processor. There are necessary transpositions (reorganisation of data) between grid point computations and fast Fourier transforms because fast Fourier transforms need complete latitudes but not necessary all the layers together.

∗ **Fast Fourier transforms:** The total number of processors involved in the A-level parallelisation is **NPRTRNS**. The total number of processors involved in the B-level parallelisation is **NPRTRV**. Fast Fourier transforms are done latitude by latitude for each **NPROMA**-packet. A processor treats a subset of latitudes, one latitude at the time. A processor can treat only a subset of the vertical levels (if **NPRTRV**>1 for inverse transforms, if **NPRTRV**>1 for direct transforms). Data reorganisation and transpositions are necessary in the Fourier space between the zonal wave structure necessary for Legendre transforms and the latitudinal structure necessary for fast Fourier transforms.

∗ **Legendre transforms:** The total number of processors involved in the A-level parallelisation is **NPRTRW**. The total number of processors involved in the B-level parallelisation is **NPRTRV**. Legendre transforms are done zonal wave number by zonal wave number. A processor treats a subset of zonal wave numbers, one zonal wave number at the time. A processor can treat only a subset of the vertical levels (if **NPRTRV**>1).

∗ **Spectral computations:** The total number of processors involved in the A-level parallelisation is **NPRTRW**. The total number of processors involved in the B-level parallelisation is **NPRTRV** for horizontal diffusion. The total number of processors involved in the B-level parallelisation is **NPRTRN** for semi-implicit scheme. Spectral computations are done zonal wave number by zonal wave number. A processor treats a subset of zonal wave numbers. For more details see documentations (IDSI) about semi-implicit scheme and (IDDH) about horizontal diffusion scheme.

∗ **Computation of Legendre polynomials:** These computation is done latitude by latitude for all the spectral coefficients. Data reorganisation and transpositions are necessary for Legendre transforms to gather all latitudes and split into zonal wave numbers because for Legendre transforms one processor treats a subset of zonal wave numbers and all latitudes.

∗ **Additional remarks about the LEQ_REGIONS environment variables.** Variables **N_REGIONS_NS**, **N_REGIONS** and **N_REGIONS_EW** are used even when **LEQ_REGIONS**=F but in this case:

- **N_REGIONS_NS=NPRGPNS**.
- **N_REGIONS=NPRGPEW** everywhere.
- **N_REGIONS_EW=NPRGPEW**.

8.2 Case LEQ_REGIONS=T.

This case is relevant only when **NPRGPEW**>1 (B-level parallelisation at least in the grid-point calculations). This is an optimised version of the LEQ_REGIONS=F case which is well designed for reduced Gaussian grid and

it improves the load balance in this case. A comprehensive description can be found in (Mozdzynski, 2006). To sum-up, we can say that:

- the A-level grid-point distribution splits the Earth into **N_REGIONS_NS** bands. **N_REGIONS_NS** can be slightly different from **NPRGPNS**.

- for each band *jroca*, the B-level grid-point distribution splits the band into **N_REGIONS**(*jroca*) zones: the minimum value of **N_REGIONS** is at the poles of the computational sphere (equal to 1 in the examples provided by Mozdzynski); the maximum value of **N_REGIONS** is at the equator of the computational sphere and this maximum is equal to **N_REGIONS_EW**. The meridian variations of **N_REGIONS** are highly correlated to those of **NLOENG**.

- In the examples provided by Mozdzynski, **NPRGPNS=NPRGPEW=NPRTRW=NPRTRV** and we notice that **N_REGIONS_NS** is slightly below **NPRGPNS**, and that **N_REGIONS_EW** is slightly below 2∗**NPRGPEW**.

9 Specific variables of the "trans" and "etrans" libraries.

Variables (and comments) can be found in the following modules of the "trans" and "etrans" libraries, see comments in the code for more details. They may have mirror counterparts in "arpifs/module".

∗ **"trans":**

- **TPM_CONSTANTS**: constants.
- **TPM_DIM**: dimensions.
- **TPM_DISTR**: variables describing distributed memory parallelization.
- **TPM_FFT**: setup for fast Fourier transforms. **TRIGS** contains the sines and cosines which are used in the matricial product which defines the Fourier transform.
- **TPM_FIELDS**: Legendre polynomials and some variables linked to the Gaussian grid. For a given total wave number *jn*, **NLTN**(*jn*) is the number of available Legendre coefficients $P(n = jn, m)$.
- **TPM_FLT**: variables for fast Legendre transforms.
- **TPM_GEN**: general control variables.
- **TPM_GEOMETRY**: data describing Gaussian grid.
- **TPM_POL**: encapsulated routines for Legendre polynomials.
- **TPM_TRANS**: variables "local" to a specific call to a transform.

∗ **"etrans":**

- **TPMALD_DIM**: dimensions.
- **TPMALD_DISTR**: variables describing distributed memory parallelization.
- **TPMALD_FFT**: setup for fast Fourier transforms. **TRIGSE** contains the sines and cosines which are used in the matricial product which defines the Fourier transform.
- **TPMALD_FIELDS**: contains for example the eigen-values of the inverse Laplace operator.
- **TPMALD_GEO**: data describing plane projection grid.
- **TPMALD_TCDIS** (useless).

10 Specific variables of the "arpifs" library.

These modules are auto-documented so description of each variable is provided in the code source. We can recall here the most important variables to know for each module:

- **YOMARG** (0-level control, former command line) and **YOMCT0** (0-level control):
    - NCONF, CUSTOP (in **NAMARG**).
    - LSPRT.
    - LRETCFOU, LWRTCFOU, LRFOUTCNORM, LRGPTCNORM (transmission coefficients in Fourier space).

    Some of these variables are in namelist **NAMCT0**.

- **YOMDIM**, **YOMDIMV** and **YOMDIMF** (dimensioning): most of variables. Some of these variables are in namelist **NAMDIM**.

- **YOMFFTTC** (base of sines and cosines used in the Fourier transforms done in CPPFTTCDIR and CPPFTTCINV).

- **MODULE_RADTC_MIX** (structures and variables for radiation transmission coefficients). In particular RFOAC and RFORADTC.

- **YOMGEM** (computational space grid-point horizontal geometry): all variables. Some of these variables are in namelist **NAMGEM**.

- **YOMLAP** (constants related to the Laplace space).

- **YOMLEG** (description of Legendre polynomials). In particular YRCSGLEG.

- **YOMMP0** and **YOMMP** (distributed memory environment, see documentation (IDDM) for more details).

- **YOMTRANS** (spectral transforms control variables). Some of these variables are in namelist **NAMTRANS** and **NAMTRANS0**.

Additional remarks:

- In a spectral array X dimensioned with **NSPEC2**, order of storing spectral coefficients is the following:
$\Re(X)_{(n=0,m=0)}$, $\Im(X)_{(n=0,m=0)}$, $\Re(X)_{(n=1,m=0)}$, $\Im(X)_{(n=1,m=0)}$, ..., $\Re(X)_{(n=N_s,m=0)}$, $\Im(X)_{(n=N_s,m=0)}$, $\Re(X)_{(n=1,m=1)}$, $\Im(X)_{(n=1,m=1)}$, $\Re(X)_{(n=2,m=1)}$, $\Im(X)_{(n=2,m=1)}$, ..., $\Re(X)_{(n=N_s,m=1)}$, $\Im(X)_{(n=N_s,m=1)}$, ..., $\Re(X)_{(n=N_s-1,m=N_s-1)}$, $\Im(X)_{(n=N_s-1,m=N_s-1)}$, $\Re(X)_{(n=N_s,m=N_s-1)}$, $\Im(X)_{(n=N_s,m=N_s-1)}$, $\Re(X)_{(n=N_s,m=N_s)}$, $\Im(X)_{(n=N_s,m=N_s)}$.

- In the array **RPNM** containing the Legendre polynomials, order of storing Legendre coefficients for a given Gaussian latitude is the following for (n,m): $(N_s+1,0)$, $(N_s,0)$, ..., $(1,0)$, $(0,0)$, $(N_s+1,1)$, $(N_s,1)$, ..., $(1,1)$, ..., (N_s+1,N_s), (N_s,N_s), ..., (N_s,N_s).

- The reader can also look at modules linked to the GMV and GFL structures:
yom_ygfl.F90, **yomgfl.F90**, **yomgmv.F90**, **type_gfls.F90**, **type_gmvs.F90**, **gfl_subs_mod.F90**, **gmv_subs_mod.F90**.
See comments inside these modules and documentation (IDEUL) for more details.

11 Bibliography.

A lot of references are available in note ARPEGE nr 30.

11.1 Publications.

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11.2 Some internal notes and other ARPEGE notes.

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- (TDECTEC) 2015: IFS technical documentation (CY41R1). Part VI: technical and computational procedures. Available at "https://software.ecmwf.int/wiki/display/IFS/Official+IFS+Documentation".
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- (IDTSAL) Radnoti, G., 2001: The transformation package for ALADIN. Internal note, 6pp.
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Appendix 1: Expression of the laplacian in spectral space (spherical geometry).

Equation:

$$\left(\nabla'^2 f\right)_{(n,m)} = \left(\frac{1}{(a\cos\Theta)^2}\frac{\partial^2 f}{\partial\Lambda^2}\right)_{(n,m)} + \left(\frac{1}{a^2\cos\Theta}\frac{\partial\left(\cos\Theta\frac{\partial f}{\partial\Theta}\right)}{\partial\Theta}\right)_{(n,m)} \tag{31}$$

can be rewritten:

$$\left((a\cos\Theta)^2\nabla'^2 f\right)_{(n,m)} = \left(\frac{\partial^2 f}{\partial\Lambda^2}\right)_{(n,m)} + \left(\cos\Theta\frac{\partial\left(\cos\Theta\frac{\partial f}{\partial\Theta}\right)}{\partial\Theta}\right)_{(n,m)} \tag{32}$$

One assumes that this equation can be rewritten as:

$$\left(\nabla'^2 f\right)_{(n,m)} = Z_{(n,m)}f_{(n,m)} \tag{33}$$

Combining equations (32) and (33) yields:

$$\left((a\cos\Theta)^2 Zf\right)_{(n,m)} = \left(\frac{\partial^2 f}{\partial\Lambda^2}\right)_{(n,m)} + \left(\cos\Theta\frac{\partial\left(\cos\Theta\frac{\partial f}{\partial\Theta}\right)}{\partial\Theta}\right)_{(n,m)} \tag{34}$$

∗ **LHS of equation (34):** One uses the following relationships:

$$(a\cos\Theta)^2 = a^2(1-\mu^2) \tag{35}$$

and:

$$[\mu X]_{(n,m)} = e_{(n,m)}X_{(n-1,m)} + e_{(n+1,m)}X_{(n+1,m)} \tag{36}$$

Applying μ to equation (36) yields:

$$[\mu^2 X]_{(n,m)} = e_{(n,m)}\left(e_{(n-1,m)}X_{(n-2,m)} + e_{(n,m)}X_{(n,m)}\right) + e_{(n+1,m)}\left(e_{(n+1,m)}X_{(n,m)} + e_{(n+2,m)}X_{(n+2,m)}\right) \tag{37}$$

i.e.

$$[\mu^2 X]_{(n,m)} = e_{(n,m)}e_{(n-1,m)}X_{(n-2,m)} + \left(e_{(n,m)}^2 + e_{(n+1,m)}^2\right)X_{(n,m)} + e_{(n+1,m)}e_{(n+2,m)}X_{(n+2,m)} \tag{38}$$

Introducing the results of equations (35) and (38) in the LHS of equation (34) yields:

$$\left((a\cos\Theta)^2 Zf\right)_{(n,m)} = \left(a^2(1-\mu^2)Zf\right)_{(n,m)} =$$

$$-a^2 e_{(n,m)}e_{(n-1,m)}[Zf]_{(n-2,m)} - a^2\left(-1 + e_{(n,m)}^2 + e_{(n+1,m)}^2\right)[Zf]_{(n,m)} - a^2 e_{(n+1,m)}e_{(n+2,m)}[Zf]_{(n+2,m)} \tag{39}$$

∗ **RHS of equation (34):** Equation (9) yields:

$$\left(\frac{\partial^2 f}{\partial\Lambda^2}\right)_{(n,m)} = -m^2 f_{(n,m)} \tag{40}$$

Equation (7) yields:

$$\left(\cos\Theta\frac{\partial\left(\cos\Theta\frac{\partial f}{\partial\Theta}\right)}{\partial\Theta}\right)_{(n,m)} =$$

$$-(n-1)e_{(n,m)}\left(-(n-2)e_{(n-1,m)}f_{(n-2,m)} + (n+1)e_{(n,m)}f_{(n,m)}\right)$$

$$+(n+2)e_{(n+1,m)}\left(-ne_{(n+1,m)}f_{(n,m)} + (n+3)e_{(n+2,m)}f_{(n+2,m)}\right) \tag{41}$$

i.e.

$$\left(\cos\Theta\frac{\partial\left(\cos\Theta\frac{\partial f}{\partial\Theta}\right)}{\partial\Theta}\right)_{(n,m)} = e_{(n,m)}e_{(n-1,m)}(n-1)(n-2)f_{(n-2,m)} +$$

$$\left(-(n-1)(n+1)e_{(n,m)}^2 - n(n+2)e_{(n+1,m)}^2\right)f_{(n,m)} + e_{(n+1,m)}e_{(n+2,m)}(n+2)(n+3)f_{(n+2,m)} \tag{42}$$

The sum of equations (40) and (42) yield:

$$\left(\frac{\partial^2 f}{\partial\Lambda^2}\right)_{(n,m)} + \left(\cos\Theta\frac{\partial\left(\cos\Theta\frac{\partial f}{\partial\Theta}\right)}{\partial\Theta}\right)_{(n,m)} = e_{(n,m)}e_{(n-1,m)}(n-1)(n-2)f_{(n-2,m)}$$

$$+\left(-\frac{m^2}{n(n+1)} - \frac{(n-1)}{n}e_{(n,m)}^2 - \frac{(n+2)}{n+1}e_{(n+1,m)}^2\right)n(n+1)f_{(n,m)} + e_{(n+1,m)}e_{(n+2,m)}(n+2)(n+3)f_{(n+2,m)} \tag{43}$$

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∗ **Identity of the LHS and the RHS:** After some strenuous calculations which are not detailed in this documentation and using the expression of $e_{(n,m)}$ (see formula (8)), one can show that the two expressions:

$$-1 + e_{(n,m)}^2 + e_{(n+1,m)}^2$$

and:

$$-\frac{m^2}{n(n+1)} - \frac{n-1}{n}e_{(n,m)}^2 - \frac{n+2}{n+1}e_{(n+1,m)}^2$$

are equal. So equation (43) can be rewritten:

$$\left(\frac{\partial^2 f}{\partial \Lambda^2}\right)_{(n,m)} + \left(\cos\Theta \frac{\partial\left(\cos\Theta \frac{\partial f}{\partial\Theta}\right)}{\partial\Theta}\right)_{(n,m)} = e_{(n,m)}e_{(n-1,m)}(n-1)(n-2)f_{(n-2,m)}$$

$$+ \left(-1 + e_{(n,m)}^2 + e_{(n+1,m)}^2\right)n(n+1)f_{(n,m)} + e_{(n+1,m)}e_{(n+2,m)}(n+2)(n+3)f_{(n+2,m)} \tag{44}$$

The identity of the LHS and RHS of equation (34) can be satisfied taking:

$$Z_{(n,m)} = -\frac{n(n+1)}{a^2} \tag{45}$$

So equation (33) can be rewritten:

$$\left(\nabla'^2 f\right)_{(n,m)} = -\frac{n(n+1)}{a^2}f_{(n,m)} \tag{46}$$

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Appendix 2: Computation of Gaussian latitudes and Gaussian weights (spherical geometry).

Appendix 2.1: First method.

Method used until CY35T2.

∗ **Gaussian latitudes:** $ndglg$ is the number of Gaussian latitudes (for convenience it is assumed to be an even number), jgl is the index numbering the latitudes. Θ is the latitude on the computational sphere, $\mu = \sin\Theta$. $\mathcal{P}_{(n,m)}$ are the unnormalized Legendre polynomials. The relationship between $P_{(n,m)}$ and $\mathcal{P}_{(n,m)}$ is:

$$\mathcal{P}_{(n,m)}(\mu) = \frac{1}{\sqrt{(2n+1)\frac{(n-m)!}{(n+m)!}}} P_{(n,m)}(\mu) \tag{47}$$

The Gaussian latitudes Θ_{jgl} match the following relationship:

$$\mathcal{P}_{(n=ndglg,m=0)}\left(\mu_{jgl}\right) = 0 \tag{48}$$

A first guess approximation for Θ_{jgl} (which are "nearly" equidistant) yields:

$$\Theta_{jgl}^{(i=0)} = \frac{\pi}{2} - \frac{4jgl-1}{4ndglg+2}\pi - \frac{1}{8ndglg^2}\frac{\cos\left(\frac{4jgl-1}{4ndglg+2}\pi\right)}{\sin\left(\frac{4jgl-1}{4ndglg+2}\pi\right)} \tag{49}$$

and thus $\mu_{jgl}^{(i=0)} = \sin\Theta_{jgl}^{(i=0)}$. An iterative Newton method is then applied to find the "final" value of μ_{jgl}.

$$\mu_{jgl}^{(i+1)} = \mu_{jgl}^{(i)} - \frac{\mathcal{P}_{(n=ndglg,m=0)}\left(\mu_{jgl}^{(i)}\right)}{\frac{d\mathcal{P}_{(n=ndglg,m=0)}}{d\mu}\left(\mu_{jgl}^{(i)}\right)} \tag{50}$$

This algorithm is used to find the Northern hemisphere latitudes (jgl from 1 to $ndglg/2$). The southern hemisphere latitudes are given by:

$$\mu_{ndglg+1-jgl} = -\mu_{jgl}$$

One can see that this algorithm needs the calculation of $\mathcal{P}_{(n=ndglg,m=0)}$ and $\frac{d\mathcal{P}_{(n=ndglg,m=0)}}{d\mu}$. This calculation is done with some recurrence formulae which will be detailed in appendix 3 for the normalized Legendre polynomials. The derivatives of the unnormalized Legendre polynomials are given by the following formula, once knowing the unnormalized Legendre polynomials:

$$\frac{d\mathcal{P}_{(n,m)}}{d\mu} = -\frac{n\mu}{1-\mu^2}\mathcal{P}_{(n,m)} + \frac{n+m}{1-\mu^2}\mathcal{P}_{(n-1,m)} \tag{51}$$

∗ **Gaussian weights:** They are given by the following formula:

$$\omega_{jgl} = \frac{1-\mu_{jgl}^2}{[ndglg * \mathcal{P}_{(n=ndglg,m=0)}\left(\mu_{jgl}\right)]^2} \tag{52}$$

Appendix 2.2: Second method.

Method used by the current version of **SUGAW**. Not yet described. Uses a method described in (Swarztrauber, 2002).

Appendix 3: Computation of the normalized Legendre polynomials (spherical geometry).

Appendix 3.1: First method.

Method used until CY35T2.

∗ **Introduction:** The algorithm uses recurrence formulae. There are several possible solutions, the following one is used in the code. The order is the following one:

- $P_{(n=0,m=0)}$; $P_{(n=1,m=0)}$; $P_{(n=1,m=1)}$.
- $P_{(n,m=0)}$ where $n > 1$.
- $P_{(n,m=1)}$ where $n > 1$.
- $P_{(n,m=n)}$ where $n > 2$.
- The remaining Legendre polynomials in the following order: $P_{(n=3,m=2)}$; $P_{(n=4,m=2)}$; $P_{(n=4,m=3)}$; $P_{(n=5,m=2)}$; $P_{(n=5,m=3)}$; $P_{(n=5,m=4)}$; ...; $P_{(n,m=2)}$; $P_{(n,m=3)}$; ...; $P_{(n,m=n-1)}$; $P_{(n+1,m=2)}$; $P_{(n+1,m=3)}$; ...; $P_{(n+1,m=n-1)}$; $P_{(n+1,m=n)}$; etc.

∗ **Wave numbers 0 and 1:**
$$P_{(n=0,m=0)} = 1$$
$$P_{(n=1,m=0)} = \sqrt{3}\mu$$
$$P_{(n=1,m=1)} = \sqrt{1.5\,(1-\mu^2)}$$

∗ **Other polynomials for** $m = 0$**:** $P_{(n,m=0)}$ is computed knowing $P_{(n-1,m=0)}$ and $P_{(n-2,m=0)}$. The following recurrence formula is used (formula (1.20) of (Rochas and Courtier, 1992)):

$$\mu P_{(n,m)} = \sqrt{(n^2-m^2)/(4n^2-1)}P_{(n-1,m)} + \sqrt{((n+1)^2-m^2)/(4(n+1)^2-1)}P_{(n+1,m)} \tag{53}$$

Changing n into $n-1$ in equation (53), then putting $P_{(n,m)}$ in the LHS, yields:

$$P_{(n,m)} = \mu\sqrt{(4n^2-1)/(n^2-m^2)}P_{(n-1,m)} - \sqrt{((n-1)^2-m^2)/(4(n-1)^2-1)}\sqrt{(4n^2-1)/(n^2-m^2)}P_{(n-2,m)} \tag{54}$$

For $m = 0$ equation (54) becomes:

$$P_{(n,m=0)} = \mu\sqrt{(4n^2-1)/n^2}P_{(n-1,m=0)} - \sqrt{(n-1)^2/(4(n-1)^2-1)}\sqrt{(4n^2-1)/n^2}P_{(n-2,m=0)} \tag{55}$$

∗ **Other polynomials for** $m = 1$**:** $P_{(n,m=1)}$ is computed knowing $P_{(n,m=0)}$ and $P_{(n-1,m=0)}$. The following recurrence formula is used (formula (1.22) of (Rochas and Courtier, 1992)):

$$\sqrt{1-\mu^2}P_{(n,m)} = \sqrt{[(2n+1)(n+m-1)]/[(2n-1)(n+m)]}P_{(n-1,m-1)} + \sqrt{(n-m+1)/(n+m)}P_{(n,m-1)} \tag{56}$$

For $m = 1$ equation (56) becomes:

$$\sqrt{1-\mu^2}P_{(n,m=1)} = \sqrt{[(2n+1)n]/[(2n-1)(n+1)]}P_{(n-1,m=0)} + \sqrt{n/(n+1)}P_{(n,m=0)} \tag{57}$$

∗ **Other polynomials for** $m = n$**:** The following recurrence formula is used:

$$P_{(n,n)} = \sqrt{1-\mu^2}\sqrt{(2n+1)/(2n)}P_{(n-1,n-1)} \tag{58}$$

∗ **Other polynomials:** The following recurrence formula is used (formula (1.31) of (Rochas and Courtier, 1992)):

$$P_{(n,m)} = \sqrt{\frac{(2n+1)(n+m-1)(n+m-3)}{(2n-3)(n+m)(n+m-2)}}P_{(n-2,m-2)} - \sqrt{\frac{(2n+1)(n+m-1)(n-m+1)}{(2n-1)(n+m)(n+m-2)}}P_{(n-1,m-2)} + \sqrt{\frac{(2n+1)(n-m)}{(2n-1)(n+m)}}P_{(n-1,m)} \tag{59}$$

Appendix 3.2: Second method.

Method used by the current version of **SUPOL**. Uses a method described in (Swarztrauber, 2002) and briefly described in (Wedi et al., 2013).

Appendix 4: Fair numbers between 1 and 50000 writing $2^{(1+p)}3^q 5^r$.

2	4	6	8	10	12	16	18	20	24	30	32	36	40	48
50	54	60	64	72	80	90	96							
100	108	120	128	144	150	160	162	180	192	200	216	240		
250	256	270	288	300	320	324	360	384	400	432	450	480	486	
500	512	540	576	600	640	648	720							
750	768	800	810	864	900	960	972							
1000	1024	1080	1152	1200	1250	1280	1296	1350	1440	1458				
1500	1536	1600	1620	1728	1800	1920	1944							
2000	2048	2160	2250	2304	2400	2430	2500	2560	2592	2700	2880	2916		
3000	3072	3200	3240	3456	3600	3750	3840	3888						
4000	4050	4096	4320	4374	4500	4608	4800	4860						
5000	5120	5184	5400	5760	5832	6000	6144	6250	6400	6480	6750	6912	7200	7290
7500	7680	7776	8000	8100	8192	8640	8748	9000	9216	9600	9720			
10000	10240	10368	10800	11250	11520	11664	12000	12150	12288					
12500	12800	12960	13122	13500	13824	14400	14580							
15000	15360	15552	16000	16200	16384	17280	17496	18000	18432	18750	19200	19440		
20000	20250	20480	20736	21600	21870	22500	23040	23328	24000	24300	24576			
25000	25600	25920	26244	27000	27648	28800	29160							
30000	30720	31104	31250	32000	32400	32768	33750	34560	34992					
36000	36450	36864	37500	38400	38880	39366								
40000	40500	40960	41472	43200	43740	45000	46080	46656	48000	48600	49152	50000		