

# The non parametrised continuous equations at a finite scale in the case of a multiphase system

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## 1 Introduction

In this paper, we consider a multiphase system of  $n+1$  components ( $k=0$  to  $n$ ). We study the content of an eulerian (geometric or fixed) volume  $V$  ( $S$  being the surrounding closed surface of  $V$ ). At a given time  $t$ , the total mass of mixture contained in the volume  $V$  is  $m$  and the mass of the species  $l$  contained in the volume  $V$  is  $m_l$  ( $m$  and  $m_l$  may vary in time).

We will also work at two different scales. The first scale is the scale of the Navier-Stokes equation, the scale of the continuum that we will also call the « local » scale. The second scale is a larger scale. It is the scale of the parametrisation in the physic part of the model or the scale represented by the truncated equations of a numerical model.

In this paper, we only give continuous formulation of the equations. The discretisation problem should be treated separately. Note also that some terms of the averaged equations will have to be parameterised in order to be estimated in a numerical model.

## 2 Definition of the averaging operators

In this section, we define the two averaging operators used to go from the scale of the continuum to a larger scale (we do not discuss the averaging operation to go from the molecular scale to the continuum scale. We start from the result of this operation).

### 2.1 Case of a monophasic system

In the case of a monophasic system, we can define two types of averaging operator :

– a volumic averaging operator such as, for any variable  $\psi$  :

$$\bar{\psi} = \frac{1}{V} \int_V \psi dv$$

The departure from the volumic average is defined as :  $\psi' = \psi - \bar{\psi}$ .

Note that with this definition :

$$\overline{\psi'} = \frac{1}{V} \int_V (\psi - \bar{\psi}) dv = \bar{\psi} - \bar{\psi} = 0$$

– a mass-weighted averaging operator such as, for any variable  $\psi$  :

$$\hat{\psi} = \frac{1}{m} \int_m \psi dm$$

with  $dm = \rho dv$  so that

$$\hat{\psi} = \frac{1}{m} \int_V \rho \psi dv$$

The departure from the mass-weighted average is defined as :  $\psi'' = \psi - \hat{\psi}$ .

Note that with this definition :

$$\widehat{\psi}'' = \frac{1}{m} \int_V \rho(\psi - \widehat{\psi})dv = \widehat{\psi} - \widehat{\psi} = 0$$

By definition also, both  $\overline{\psi}$  and  $\widehat{\psi}$  are uniform value in the volume  $V$  (a practical consequence is that they may be taken out of the integral sign in the averaging operators).

We also know that

$$m = \int_V \rho dv$$

hence, using the volumic averaging, we have :

$$\overline{\rho} = \frac{1}{V} \int_V \rho dv = \frac{m}{V} \text{ or } m = \overline{\rho}V$$

We can note that :

$$\widehat{\psi} = \frac{1}{\overline{\rho}V} \int_V \rho\psi dv = \frac{\overline{\rho\psi}}{\overline{\rho}} \text{ or } \overline{\rho\psi} = \overline{\rho}\widehat{\psi}$$

and (see details in box 1)

$$\overline{\rho\psi\eta} = \overline{\rho}\widehat{\psi}\widehat{\eta} + \overline{\rho\psi''\eta''}$$

$$\begin{aligned} \overline{\rho\psi\eta} &= \frac{1}{V} \int_V (\rho\psi\eta)dv \\ &= \overline{\rho} \frac{1}{m} \int_V (\rho\psi\eta)dv \\ &= \overline{\rho} \frac{1}{m} \int_V \rho([\widehat{\psi} + \psi''][\widehat{\eta} + \eta''])dv \\ &= \overline{\rho} \frac{1}{m} \int_V \rho(\widehat{\psi}\widehat{\eta})dv + \overline{\rho} \frac{1}{m} \int_V \rho(\widehat{\psi}\eta'')dv + \overline{\rho} \frac{1}{m} \int_V \rho(\psi''\widehat{\eta})dv + \frac{1}{V} \int_V \rho(\psi''\eta'')dv \\ &= \overline{\rho}\widehat{\psi}\widehat{\eta} + 0 + 0 + \overline{\rho\psi''\eta''} \end{aligned}$$

**Box 1:**

## 2.2 Case of a multiphase system

In the case of a multiphase system, we have to introduce an additional discrete (mass-weighted or barycentric) averaging operator to evaluate the local variables for the mixture :

$$\psi = \sum_k \frac{\rho_k}{\rho} \psi_k = \sum_k q_k \psi_k$$

where

$$\rho = \sum_k \rho_k$$

and

$$q_k = \frac{\rho_k}{\rho}$$

The local departure between the mixture value and the value of the species  $l$  is defined as :

$$\tilde{\psi}_l = \psi_l - \psi$$

Note that

$$\sum_k (\rho_k \tilde{\psi}_k) = 0$$

The volumic or mass-weighted operator defined in the previous section may be applied to the local mixture variables.

The volumic average of any mixture variable  $\psi$  is given by :

$$\bar{\psi} = \frac{1}{V} \int_V \psi dv$$

But we can also compute the volumic average of the variable  $\psi_l$  characteristic of the species  $l$  by :

$$\bar{\psi}_l = \frac{1}{V} \int_V \psi_l dv$$

Then, the total mass of the parcel is given by :

$$m = \int_V (\sum_k \rho_k) dv = \int_V \rho dv$$

and the averaged density of the mixture is :

$$\bar{\rho} = \frac{1}{V} \int_V (\sum_k \rho_k) dv = \frac{m}{V}$$

The total mass of the species  $l$  in the volume  $V$  is given by :

$$m_l = \int_V \rho_l dv$$

and the averaged density of the species  $l$  in the volume  $V$  is :

$$\bar{\rho}_l = \frac{1}{V} \int_V \rho_l dv = \frac{m_l}{V}$$

The mass-weighted averaging in the multiphase case may be defined as a mass-weighted averaging with respect to the mixture mass of the parcel such as :

$$\widehat{\psi} = \frac{1}{m} \int_m \psi dm = \frac{1}{m} \int_V \rho \psi dv$$

with  $dm = \rho dv$  where  $\rho = \sum_k \rho_k$ .

The departure  $\psi'' = \psi - \widehat{\psi}$  has the nice property of having a mass-weighted averaged value equal to zero ( $\widehat{\psi''} = 0$ , «centred» variable).

We can also compute the barycentric average of the variable  $\psi_l$  by :

$$\widehat{\psi}_l = \frac{1}{m} \int_m \psi_l dm = \frac{1}{m} \int_V \rho \psi_l dv$$

The departure  $\psi_l'' = \psi_l - \widehat{\psi}_l$  has also the nice property of having a mass-weighted averaged value equal to zero ( $\widehat{\psi_l''} = 0$ ).

Usually, we are also interested by the departure between the local variable  $\psi_l$  and the mass averaged mixture value  $\widehat{\psi}$  defined by :

$$\psi_l = \widehat{\psi} + \psi_l'''$$

But, in that case, the departure between the variable  $\psi_l$  and the averaged mixture value  $\widehat{\psi}$  has NOT the nice property of having an averaged mixture value equal to zero. Actually (see box 2),

$$\widehat{\psi_l'''} = \widehat{\psi}_l - \widehat{\psi} = \widehat{\psi}_l$$

This « bad » property of  $\psi_l'''$  will have some important consequences on the conservation equation developed below.

$$\begin{aligned}
\widehat{\psi}_l''' &= \frac{1}{m} \int_V \rho \psi_l''' dv = \frac{1}{m} \int_V \rho (\psi_l - \widehat{\psi}) dv \\
&= \frac{1}{m} \int_V \rho \psi_l dv - \frac{1}{m} \int_V \rho \widehat{\psi} dv \\
&= \widehat{\psi}_l - \widehat{\psi} = \widehat{\psi}_l - \widehat{\psi} = \widehat{\psi}_l
\end{aligned}$$

**Box 2:**

### 3 Equation of state

#### 3.1 Case of a monophasic gaz

For a sake of better understanding the problem of the averaging of a non linear diagnostic equation like the perfect gaz state equation, we will first treat it in the simple case of a monophasic perfect gaz (it will also be valid in the case of a gaz mixture with constant and uniform concentrations).

At the local scale, the equation of state is written

$$p = \rho R_{gaz} T \quad (1)$$

where  $R_{gaz} = R^*/M_{gaz}$ ,  $R^*$  is the perfect gaz constant ( $R^* = 8,314 \text{ J.mol}^{-1}.\text{K}^{-1}$ ) and  $M_{gaz}$  the molar mass of the gaz.

Applying the volumic average to this equation, we get :

$$\bar{p} = \overline{\rho R_{gaz} T} \quad (2)$$

As, in this context,  $R_{gaz}$  is constant, this equation may be transformed to :

$$\bar{p} = R_{gaz} \overline{\rho T} = R_{gaz} \bar{\rho} \widehat{T} \quad (3)$$

The use of the mass-weighted average allows the conservation of the classical form of the perfect gaz state equation for the averaged state variables  $\bar{p}$ ,  $\bar{\rho}$  and  $\widehat{T}$ <sup>1</sup>

#### 3.2 Case of a multiphase mixture

**H1** : In this paper, we do the hypothesis that all the species which are present in the parcel are instantaneously in local thermic equilibrium, so all the species in the parcel have the same local temperature :  $T_k = T$  for  $k = 0, n$  (this hypothesis is not done in Bannon, 2002, but we do it here INTRODUIRE ICI LA REMARQUE DE JF SUR LA TEMPERATURE D'EQUILIBRE).

Even in the case of a multiphase mixture, the « equation of state » is an equation written for the state of the gaz species only.

The local form of the state equation for dry air (k=0) and wator vapor (k=1) are :

$$\begin{aligned}
p_0 &= \rho_0 R_0 T \\
p_1 &= \rho_1 R_1 T
\end{aligned}$$

<sup>1</sup>Using only volumic averages would give :

$$\bar{p} = R_{gaz} \overline{\rho T} = R_{gaz} \bar{\rho} \widehat{T} + R_{gaz} \overline{\rho' T'} \quad (4)$$

To conserve the classical form of the state equation, we would have to neglect the second term on the right hand side of this equation.

Applying the volumic averaging operator to these equations gives :

$$\begin{aligned}\bar{p}_0 &= \overline{\rho_0 R_0 T} \\ \bar{p}_1 &= \overline{\rho_1 R_1 T}\end{aligned}$$

But, the densities in these equations are not the density of the mixture which is used to define the mass-weighted operator. However, we can transform a bit these equations to make  $\rho$  appear :

$$\begin{aligned}\bar{p}_0 &= \overline{\rho q_0 R_0 T} \\ \bar{p}_1 &= \overline{\rho q_1 R_1 T}\end{aligned}$$

From these last two equations, we finally obtain :

$$\begin{aligned}\bar{p}_0 &= \bar{\rho} \hat{q}_0 R_0 \hat{T} + R_0 \overline{\rho q_0'' T''} \\ \bar{p}_1 &= \bar{\rho} \hat{q}_1 R_1 \hat{T} + R_1 \overline{\rho q_1'' T''}\end{aligned}$$

**H2** : To keep the classical form of the state equation for the averaged variables, we have to neglect the last terms on the right hand side of these two equations :

$$\begin{aligned}\bar{p}_0 &= \bar{\rho} \hat{q}_0 R_0 \hat{T} \\ \bar{p}_1 &= \bar{\rho} \hat{q}_1 R_1 \hat{T}\end{aligned}$$

With such hypothesis, the Dalton's law is still valid for the mean quantities and then :

$$\bar{p} = \bar{p}_0 + \bar{p}_1 = \bar{\rho} (R_0 \hat{q}_0 + R_1 \hat{q}_1) \hat{T}$$

EXPLORER LA PROPOSITION DE JF (R NULS POUR LES CONDENSATS)

## 4 The momentum of a mixture

In this paper, we suppose that the local horizontal wind is the same for all the species which are present in the mixture. Somehow, we suppose then that all the species are instantaneously in some kind of local horizontal mechanic equilibrium. Note that this hypothesis is very similar to the local thermodynamic equilibrium hypothesis we have done previously (section 1.2.2).

Let's  $u$  be the local zonal component and  $v$  the local meridional component of the horizontal wind of the mixture. With the mechanic equilibrium hypothesis,  $u$  and  $v$  are also the horizontal component of the local wind of any species  $l$  present in the mixture ( $u_l = u$  and  $v_l = v$ ).

However, if the mixture contains some precipitating species which are falling with different speeds, the vertical velocities of the different species are not the same.

The local vertical velocity of the mixture  $w$  (or local barycentric vertical velocity) is such as :

$$\rho w = \sum_k \rho_k w_k$$

where  $w_k$  is the local vertical velocity of the species  $k$ .

$\tilde{w}_k = w_k - w$  is the local departure between the vertical velocity of the species  $k$  and the mixture vertical velocity. Note that :

$$\sum_k \rho_k \tilde{w}_k = 0$$

The relative (with respect to the barycentric velocity) vertical velocity of the precipitating species is negative. We define *the mean relative velocity of the precipitating species* by :

$$\tilde{w}_{precip} = \frac{1}{\rho_{precip}} \sum_{precip} \rho_k \tilde{w}_k$$

Similarly, we define the relative velocity of the non precipitating species (**H4** :we suppose here that all the non precipitating species have the same relative vertical velocity) by :

$$\tilde{w}_{np} = \frac{1}{\rho_{np}} \sum_{np} \rho_k \tilde{w}_k = -\frac{\rho_{precip}}{\rho_{np}} \tilde{w}_{precip}$$

The relative vertical velocity of the non precipitating species is then positive (relative upward motion).

The average vertical velocity of the species  $l$  is defined as :

$$\widehat{w}_l = \frac{1}{\bar{\rho}V} \int_V \rho w_l dv$$

and the average vertical velocity of the mixture is defined as :

$$\widehat{w} = \frac{1}{\bar{\rho}V} \int_V \rho w dv$$

We also have :

$$w_l = \widehat{w}_l + w_l'' = \widehat{w} + w_l'''$$

and

$$w = \widehat{w} + w''$$

We check that :

$$\widehat{w_l''} = 0$$

and

$$\widehat{w''} = 0$$

but

$$\widehat{w_l'''} = \widehat{w}_l - \widehat{w} = \widehat{w}_l$$

We will see later that  $\widehat{w}_l$  is a function of the fall velocity of the different species and that it is at the basis of the definition of the precipitation fluxes.

Note also that, for any species  $l$  in the mixture,  $\tilde{u}_l = \tilde{v}_l = 0$

## 5 The total mass conservation in a multiphase mixture

In this section we are looking for an equation describing the total mass evolution of the mixture in the Eulerian volume  $V$ . The mass of mixture in the volume  $V$  varies because of the total contribution of the different mass flux in/out the volum  $V$  and because sources of each of the species in this volume :

$$\frac{\partial m}{\partial t} = \frac{\partial (\int_V \rho dv)}{\partial t} = - \int_S \sum_k (\rho_k \vec{u}_k \cdot \vec{n}) dS + \int_V \sum_k \dot{\rho}_k dv \quad (5)$$

where  $m = \bar{\rho}V = \int_V \rho dv$  and  $\dot{\rho}_k$  is the volumic source of the species  $k$ .

We know that in the atmosphere  $\sum_k \dot{\rho}_k = 0$ , hence :

$$\frac{\partial m}{\partial t} = - \int_S \sum_k (\rho_k \vec{u}_k \cdot \vec{n}) dS \quad (6)$$

Using the Ostrogradsky theorem for the right hand side, this equation may also be written :

$$\frac{\partial m}{\partial t} = \int_V \frac{\partial(\rho)}{\partial t} dv = - \int_V \text{div}(\sum_k \rho_k \vec{u}_k) dv = - \int_V \text{div}(\rho \vec{u}) dv \quad (7)$$

## 5.1 Local form of the continuity equation

To obtain the local form of the continuity equation, the classic way is to say that :

- as the volum  $V$  is fixed in time, the integral and the time derivative commutes in the left hand side. After the commutation operation, the volumic integral operator is then « outside » for each of the terms in this equation ;
- as the « integral » form of the equation is valid for any volume  $V$ , it is valid locally everywhere.

The local form of the continuity equation in the multiphase case conserves a form we classically use in the monophasic case :

$$\frac{\partial \rho}{\partial t} = -\text{div}(\rho \vec{u}) \quad (8)$$

but with  $\vec{u} = \sum_k q_k \vec{u}_k$ , the local « barycentric » velocity of the mixture.

## 5.2 Average form of the continuity equation

The equation 1.7 may also be written as :

$$\frac{\partial \bar{\rho} V}{\partial t} = - \int_V \text{div}(\rho \vec{u}) dv = -V \text{div} \left( \frac{1}{V} \int_V \rho \vec{u} dv \right) \quad (9)$$

Then, the average form of the continuity equation also conserves the classical « Eulerian » form of the continuity equation :

$$\frac{\partial \bar{\rho}}{\partial t} = -\text{div}(\bar{\rho} \vec{u}) = -\text{div}(\widehat{\rho} \vec{u}) \quad (10)$$

This equation may also be written in a « barycentric » Lagrangian form (with an advection by the average velocity of the mixture) :

$$\frac{\widehat{D} \bar{\rho}}{Dt} = \frac{\partial \bar{\rho}}{\partial t} + \widehat{u} \cdot \text{grad}(\bar{\rho}) = -\bar{\rho} \text{div}(\widehat{u}) \quad (11)$$

## 6 General budget equation in a multiphase system

In the former section, we have done the mass budget in an eulerian volum  $V$ . The result of this mass budget was useful to deduce a local mass equation and an adverage mass equation. A similar method will be use for the mass of each species  $l$  of the mixture, for the total momentum and for the total energy, but, in this section, we will expose in a more general way the budget of a specific quantity (quantity by mass unit)  $\psi$ .

The quantity  $\psi$  associated with the species  $l$  is  $\psi_l$ . The quantity of the mixture is defined with a mass-weighted average such as  $\rho \psi = \sum_k \rho_k \psi_k$ .

The budget of the total quantity of  $\psi$  in any Eulerian volume  $V$  (volume fixed in time) is :

$$\underbrace{\frac{\partial \int_V \sum_k (\rho_k \psi_k) dv}{\partial t}}_A = - \underbrace{\int_S \sum_k (\rho_k \psi_k \vec{u}_k \cdot \vec{n}) ds}_B + \underbrace{\int_V \sum_k \dot{S}_k dv}_C \quad (12)$$

where

**term A** is the eulerian evolution of the budget of the total  $\psi$  in the volume  $V$

**term B** is the budget of the fluxes of  $\psi_k$  through the surrounding surface of the volume  $V$

**term C** is the budget of the volumic sources  $\dot{S}_k$  of  $\rho_k \psi_k$ .

Using the Ostrogradsky theorem, and commuting the time derivative and the volumic integral in the left hand side, this equation becomes :

$$\int_V \frac{\partial (\rho \psi)}{\partial t} dv = - \int_V \text{div} \left[ \sum_k (\rho_k \psi_k \vec{u}_k) \right] dv + \int_V \sum_k \dot{S}_k dv \quad (13)$$

## 6.1 Equation for the local value of $\psi$

As this integral form of the equation for  $\psi$  is valid for any volum  $V$ , it is valid locally for the integrand. This gives the local form of the equation for  $\psi$  :

$$\frac{\partial(\rho\psi)}{\partial t} = -\text{div}\left[\sum_k (\rho_k\psi_k\vec{u}_k)\right] + \sum_k \dot{S}_k \quad (14)$$

For each species  $k$ , the velocity may be decomposed in  $\vec{u}_k = \vec{u} + \tilde{\vec{u}}_k$ . The former equation becomes :

$$\frac{\partial(\rho\psi)}{\partial t} = -\text{div}(\rho\psi\vec{u}) - \text{div}\left[\sum_k (\rho_k\psi_k\tilde{\vec{u}}_k)\right] + \sum_k \dot{S}_k \quad (15)$$

Using the continuity equation and the hypothesis of horizontal homogeneity for the horizontal velocity of the different species (**H3**, see section 1.3), this can also be written as :

$$\underbrace{\rho \frac{\partial(\psi)}{\partial t}}_A + \underbrace{\rho \vec{u} \cdot \text{grad}(\psi)}_B = - \underbrace{\frac{\partial[\sum_k (\rho_k\psi_k\tilde{w}_k)]}{\partial z}}_C + \underbrace{\sum_k \dot{S}_k}_D \quad (16)$$

where

**term A** is the eulerian evolution of  $\psi$

**term B** is minus the advection of  $\psi$  by the local barycentric velocity

**term C** is the budget of the transports of  $\psi_k$  by the departure with respect to the barycentric velocity

**term D** is the budget of the volumic sources of  $\psi_k$ .

## 6.2 Equation for the average value of $\psi$

The equation 1.13 is also useful to derive an equation for the averaged value of  $\psi$  in a finite volume  $V$ . With a few manipulation (let's remind that  $V$  is a constant in time and space), we get :

$$\frac{\partial \bar{\rho} \int_V (\rho\psi) dv}{\partial t} = -\text{div}\left[\frac{1}{V} \int_V (\rho\psi\vec{u}) dv\right] - \text{div}\left[\sum_k \frac{1}{V} \int_V (\rho_k\psi_k\tilde{\vec{u}}_k) dv\right] + \sum_k \left[\frac{1}{V} \int_V \dot{S}_k dv\right] \quad (17)$$

Using the definition of the averaging operator of section 1.1, we can write :

$$\frac{\partial(\bar{\rho}\hat{\psi})}{\partial t} = -\text{div}(\overline{\rho\psi\vec{u}}) - \frac{\partial[\sum_k \overline{\rho_k\psi_k\tilde{w}_k}]}{\partial z} + \sum_k \bar{\dot{S}}_k \quad (18)$$

After decomposition of the different variables in the sum of an averaged and a « turbulent » part, we obtain :

$$\begin{aligned} \frac{\partial(\bar{\rho}\hat{\psi})}{\partial t} &= -\text{div}(\overline{\rho\psi\vec{u}}) - \text{div}(\overline{\rho\psi''\vec{u}''}) \\ &\quad - \frac{\partial[\sum_k \overline{\rho\hat{q}_k\hat{\psi}_k\hat{w}_k}]}{\partial z} - \frac{\partial[\sum_k \overline{\hat{q}_k\rho\psi_k''\tilde{w}_k''}]}{\partial z} - \frac{\partial[\sum_k \overline{\hat{\psi}_k\rho q_k''\tilde{w}_k''}]}{\partial z} - \frac{\partial[\sum_k \overline{\hat{w}_k\rho q_k''\psi_k''}]}{\partial z} - \frac{\partial[\sum_k \overline{\rho q_k''\psi_k''\tilde{w}_k''}]}{\partial z} \\ &\quad + \sum_k \bar{\dot{S}}_k \end{aligned}$$

The second term in the right hand side is the vertical budget of the vertical turbulent fluxes of  $\psi$ . This term is usually computed in the turbulent scheme in a NWP model.



**H5** : Between the 5 terms of the decomposition of the transport by the departures with respect to the barycentric velocity, we will conserve only the first one (and neglect the four other ones).

So finally, the average form of the  $\psi$  equation is :

$$\frac{\partial(\overline{\rho\psi})}{\partial t} = -\text{div}(\overline{\rho\psi\vec{u}}) - \text{div}(\overline{\rho\psi''\vec{u}''}) - \frac{\partial[\sum_k \overline{\rho q_k \psi_k \hat{w}_k}]}{\partial z} + \sum_k \overline{S_k} \quad (19)$$

## 7 The conservation of one of the species in a multiphase mixture

We are now looking for an equation for the evolution of the mass of one of the species of the mixture.

For the species  $l$ , we can write :

$$\frac{\partial m_l}{\partial t} = - \int_S \rho_l \vec{u}_l \cdot \vec{n} dS + \int_V \dot{\rho}_l dv \quad (20)$$

with  $m_l = \overline{\rho_l} V = \int_V \rho_l dv = \int_V \rho_l q_l dv$ .

Using the Ostrogradsky theorem for the first term on the right hand side of the last equation, we get :

$$\frac{\partial m_l}{\partial t} = - \int_V \text{div}(\rho_l \vec{u}_l) dv + \int_V \dot{\rho}_l dv \quad (21)$$

### 7.1 Local form

Commuting the time derivative and the integral on the left hand side, and generalising the result to any volum  $V$ , we write :

$$\frac{\partial \rho_l}{\partial t} = -\text{div}(\rho_l \vec{u}_l) + \dot{\rho}_l \quad (22)$$

The local velocity of the species  $l$  is the sum of the barycentric velocity  $\vec{u}$  and a departure with respect to the local barycentric velocity  $\vec{w}_l$ .

Equation 1.22 becomes :

$$\frac{\partial \overbrace{\rho_l}^{\rho q_l}}{\partial t} = -\text{div}(\rho q_l \vec{u}) - \frac{\partial \rho q_l \vec{w}_l}{\partial z} + \dot{\rho}_l \quad (23)$$

In practice, we rather use an equation for the specific concentration  $q_l$  (FAIRE AUSSI LA VERSION RAPPORT DE MELANGE) :

$$\rho \frac{\partial q_l}{\partial t} + \rho \vec{u} \cdot \vec{\text{grad}} q_l = - \frac{\partial \rho q_l \vec{w}_l}{\partial z} + \dot{\rho}_l \quad (24)$$

### 7.2 Average form

Equation 1.21 may also be written as :

$$\frac{\partial \overline{\rho_l}}{\partial t} = - \frac{1}{V} \int_V \text{div}(\rho_l \vec{u}_l) dv + \overline{\dot{\rho}_l} \quad (25)$$

$$= -\text{div}\left(\frac{1}{V} \int_V \rho_l \vec{u}_l dv\right) + \overline{\dot{\rho}_l} \quad (26)$$

We already defined :

$$\widehat{q}_l = \frac{1}{m} \int_V \rho q_l dv = \frac{1}{m} \int_V \rho_l dv = \frac{m_l}{m} = \frac{\bar{\rho}_l}{\bar{\rho}}$$

and

$$\begin{aligned} \vec{u}_l &= \widehat{u} + \vec{u}_l''' \\ q_l &= \widehat{q}_l + q_l'' \end{aligned}$$

Introducing these definitions in the first right hand side term of equation 1.26, we obtain :

$$\begin{aligned} \frac{1}{V} \int_V \rho_l \vec{u}_l dv &= \frac{1}{V} \int_V \rho (\widehat{q}_l + q_l'') (\widehat{u} + \vec{u}_l''') dv \\ &= \frac{1}{V} \int_V \rho \widehat{q}_l \widehat{u} dv + \frac{1}{V} \int_V \rho \widehat{q}_l \vec{u}_l''' dv + \frac{1}{V} \int_V \rho q_l'' \widehat{u} dv + \frac{1}{V} \int_V \rho q_l'' \vec{u}_l''' dv \\ &= \overline{\rho \widehat{q}_l \widehat{u}} + \widehat{q}_l \frac{1}{V} \int_V \rho \vec{u}_l''' dv + 0 + \overline{\rho q_l'' \vec{u}_l'''} \\ &= \overline{\rho \widehat{q}_l \widehat{u}} + \widehat{q}_l \overline{\rho \vec{u}_l'''} + \overline{\rho q_l'' \vec{u}_l'''} \\ &= \overline{\rho \widehat{q}_l \widehat{u}} + \overline{\rho \widehat{q}_l \vec{u}_l} + \overline{\rho q_l'' \vec{u}_l'''} \end{aligned}$$

So finally, equation 1.26 becomes :

$$\frac{\partial \overline{\rho \widehat{q}_l}}{\partial t} = -\text{div}(\overline{\rho \widehat{q}_l \widehat{u}}) - \text{div}(\overline{\rho \widehat{q}_l \vec{u}_l}) - \text{div}(\overline{\rho q_l'' \vec{u}_l'''}) + \bar{\rho}_l \quad (27)$$

This equation may also be written as :

$$\frac{\widehat{D} \widehat{q}_l}{Dt} = \bar{\rho} \left( \frac{\partial \widehat{q}_l}{\partial t} + \widehat{u} \cdot \text{grad}(\widehat{q}_l) \right) = -\frac{\partial (\overline{\rho \widehat{q}_l \widehat{u}_l})}{\partial z} - \frac{\partial (\overline{\rho q_l'' w_l'''})}{\partial z} + \bar{\rho}_l \quad (28)$$

The first term on the right hand side of this equation is not present in the monophasic case. This term is a vertical transport of the species  $l$  by the departure between the mean (in the sens of the « hat » operator) velocity of the species  $l$  and the mixture mean vertical velocity. This term has a different nature compared to the one of the pure diffusion term  $-\partial (\overline{\rho q_l'' w_l'''}) / \partial z$  because we suppose that  $\widehat{w}_l$  is (more or less) solved at the scale of the parametrisation (with some hypothesis on the fall speed), or at least, these fluxes do not obey the property of the diffusion turbulent fluxes.

## 8 The equation for the total momentum of a mixture

The total momentum in the Eulerian volume  $V$  containing a multiphase mixture is varying because of momentum fluxes of the different species and because of forces acting on the mixture. We usually distinguish two type of forces, the volumic (or massic) forces which are proportionnal to the mass of material (gravity, Coriolis) and the « surfacic » forces which are molecular scale interaction (momentum transferts of molecular scale) at the border of the parcel (pressure and viscous forces).

**H6** : The possible momentum transfert which occur during shocks and interactions between drops, ice crystals etc at the border of the parcel are neglected here.

The budget of the component  $\alpha$  of the total momentum in the volume  $V$  may then be written as :

$$\frac{\partial (\int_V (\sum_k \rho_k u_k^\alpha) dv)}{\partial t} = - \int_S \left( \sum_k \rho_k u_k^\alpha \vec{u}_k \right) \cdot \vec{n} ds \quad (29)$$

$$+ \underbrace{\int_V \left( \sum_k \rho_k g^\alpha \right) dv + \int_V \left( \sum_k \rho_k [2\vec{\Omega} \wedge \vec{u}_k]^\alpha \right) dv + \int_S [\vec{\tau}_s \cdot \vec{n}]^\alpha ds}_{\text{sources}}$$

where the first term on the right hand side is the budget of input/output of momentum across the surface around the volume  $V$ , the second term is the gravity, the third term is the Coriolis force and the last term is the contribution of the stress tensor.

The stress tensor is defined such as  $\vec{\tau}_s \cdot \vec{n} ds$  is the surfacic force acting on a surface element  $ds$  perpendicular to the unit vector  $\vec{n}$  ( $[\vec{\tau}_s \cdot \vec{n}]^\alpha$  is the component along the  $\alpha$  axis of this force).

**H7** : We suppose here that  $\vec{\tau}_s$  may be written as the sum of two tensors :

$$\vec{\tau}_s = -p \vec{\delta} + \vec{\sigma}$$

where  $\vec{\delta}$  is the « identity » tensor,  $p$  is the total pressure of the gaz present in the mixture ( $p = p_0 + p_1$ ) and  $\vec{\sigma}$  is a symmetric tensor called the viscous stress tensor (usually this term is neglected in meteorology, except if the turbulence is closed with a TKE equation. In that case, this term may be parameterised through the energy cascade in the TKE equation).

With this hypothesis, equation 1.29 becomes :

$$\begin{aligned} \int_V \left[ \frac{\partial (\sum_k \rho_k u_k^\alpha)}{\partial t} \right] dv &= - \int_V \text{div} \left( \sum_k \rho_k u_k^\alpha \vec{u}_k \right) dv \\ &+ \int_V \left( \sum_k \rho_k g^\alpha \right) dv + \int_V \left( \sum_k \rho_k [2\vec{\Omega} \wedge \vec{u}_k]^\alpha \right) dv - \int_V [\text{grad}(p)]^\alpha dv + \int_V \text{div}[\vec{\sigma}_s^\alpha] dv \end{aligned} \quad (30)$$

Finally, this equation can also be written as :

$$\begin{aligned} \int_V \left[ \frac{\partial (\rho u^\alpha)}{\partial t} \right] dv &= - \int_V \text{div} \left( \sum_k \rho_k u_k^\alpha \vec{u}_k \right) dv \\ &+ \int_V (\rho g^\alpha) dv + \int_V (\rho [2\vec{\Omega} \wedge \vec{u}]^\alpha) dv - \int_V [\text{grad}(p)]^\alpha dv + \int_V \text{div}[\vec{\sigma}_s^\alpha] dv \end{aligned} \quad (31)$$

To go further, we will treat separately the horizontal and the vertical components of the wind.

## 8.1 Horizontal momentum conservation

In order to simplify the notations, we adopt the reduced expression of the Coriolis form valid with the « thin shell » approximation. In that case, the equation 1.31 for  $\alpha = 1$  becomes :

$$\int_V \left[ \frac{\partial (\rho u)}{\partial t} \right] dv = - \int_V \text{div} \left( \sum_k \rho_k u_k \vec{u}_k \right) dv + \int_V [2\rho\Omega \sin(\varphi)v] dv - \int_V \left[ \frac{\partial (p)}{\partial x} \right] dv + \int_V [\text{div}(\sigma_u)] dv \quad (32)$$

With the hypothesis of horizontal mechanic equilibrium, the first term on the right hand side simplifies :

$$\begin{aligned} - \int_V \text{div} \left( \sum_k \rho_k u \vec{u}_k \right) dv &= \int_V \text{div} \left( \left( \sum_k \rho_k \vec{u}_k \right) u \right) dv \\ &= \int_V \text{div}(\rho \vec{u} u) dv \end{aligned}$$

### 8.1.1 Local horizontal momentum equations

As the equation 1.32 is valid for any eulerian volum  $V$ , then :

$$\frac{\partial(\rho u)}{\partial t} = -\text{div}(\rho u \vec{u}) + 2\rho\Omega \sin(\varphi)v - \frac{\partial p}{\partial x} + \text{div}\sigma_u \quad (33)$$

or, after combination with the local continuity equation :

$$\rho \left[ \frac{\partial(u)}{\partial t} + \vec{u} \cdot \vec{\text{grad}}(u) \right] = 2\rho\Omega \sin(\varphi)v - \frac{\partial p}{\partial x} + \text{div}\sigma_u \quad (34)$$

With a similar way, we obtain an equation for the local meridional component :

$$\rho \left[ \frac{\partial(v)}{\partial t} + \vec{u} \cdot \vec{\text{grad}}(v) \right] = -2\rho\Omega \sin(\varphi)u - \frac{\partial p}{\partial y} + \text{div}\sigma_v \quad (35)$$

### 8.1.2 Average horizontal momentum equations

From equation 1.32, we can also obtain an equation for the averaged zonal momentum :

$$\frac{\partial(\int_V [\rho u]) dv}{\partial t} = -\text{div} \left( \int_V \rho u \vec{u} dv \right) + 2\Omega \sin(\varphi) \int_V (\rho v) dv - \frac{\partial(\int_V (p) dv)}{\partial x} + \text{div} \left( \int_V (\sigma_u) dv \right) \quad (36)$$

Using the average operators definition, we get :

$$\frac{\partial(\overline{\rho \hat{u}})}{\partial t} = -\text{div}(\overline{\rho \hat{u} \hat{u}}) - \text{div}(\overline{\rho u'' u''}) + 2\overline{\rho} \Omega \sin(\varphi) \hat{v} - \frac{\partial \overline{p}}{\partial x} + \text{div}(\overline{\sigma_u}) \quad (37)$$

or also :

$$\overline{\rho} \frac{\hat{D}\hat{u}}{Dt} = \overline{\rho} \left[ \frac{\partial \hat{u}}{\partial t} + \hat{u} \cdot \vec{\text{grad}}(\hat{u}) \right] = -\text{div}(\overline{\rho u'' u''}) + 2\overline{\rho} \Omega \sin(\varphi) \hat{v} - \frac{\partial \overline{p}}{\partial x} + \text{div} \overline{\sigma_u} \quad (38)$$

where  $-\hat{u} \cdot \vec{\text{grad}}(\hat{u})$  is the advection of the average zonal wind of the mixture by the average 3D wind of the mixture.

## 8.2 Vertical momentum conservation

The form of the vertical momentum equation depends on the size of the scale of the parametrisation. For parametrisation scale such as the horizontal scale is much larger than the vertical scale, we can do the hydrostatic approximation, but for horizontal scale smaller than the vertical one, we need a pronostic equation for the vertical velocity. This equation has a more complex form when the parcel is a mixture with precipitating species.

### 8.2.1 The hydrostatic case

In the hydrostatic case, the vertical momentum equation is reduced to the equilibrium between the gravity and the vertical component of the pressure force budget.

At the local scale, the hydrostatic equation is :

$$0 = -\rho g - \frac{\partial p}{\partial z}$$

The averaged form of this equation is :

$$0 = -\overline{\rho} g - \frac{\partial \overline{p}}{\partial z}$$

### 8.2.2 The non hydrostatic case

In the non hydrostatic case, we have to develop the complete form of the vertical momentum equation.

The local form of the NH vertical momentum equation is derived from equation 1.31 in the case  $\alpha = 3$  :

$$\int_V \left[ \frac{\partial(\rho w)}{\partial t} \right] dv = - \int_V \operatorname{div} \left( \sum_k \rho_k w_k \vec{u}_k \right) dv - \int_V (\rho g) dv - \int_V \left[ \frac{\partial(p)}{\partial z} \right] dv + \int_V \operatorname{div}(\sigma_w) dv \quad (39)$$

The local form becomes, after simplification :

$$\frac{\partial(\rho w)}{\partial t} = -\operatorname{div}(\rho w \vec{u}) - \frac{\partial(\sum_k \rho_k \tilde{w}_k \tilde{u}_k)}{\partial z} - \rho g - \frac{\partial(p)}{\partial z} + \operatorname{div}(\sigma_w) \quad (40)$$

The average form of the NH equation may be derived from equation 1.39 :

$$\frac{\partial(\int_V \rho w dv)}{\partial t} = -\operatorname{div} \left( \int_V \sum_k \rho_k w_k \vec{u}_k dv \right) - g \int_V \rho dv - \frac{\partial(\int_V p dv)}{\partial z} + \operatorname{div} \left( \int_V \sigma_w dv \right) \quad (41)$$

Using the definition of the averaging operators, we get :

$$\frac{\partial(\bar{\rho} \hat{w})}{\partial t} = -\operatorname{div}(\bar{\rho} \hat{w} \hat{\vec{u}}) - \operatorname{div}(\overline{\rho w'' \vec{u}''}) - \frac{\partial(\sum_k \overline{\rho_k \tilde{w}_k^2})}{\partial z} - \bar{\rho} g - \frac{\partial \bar{p}}{\partial z} + \operatorname{div}(\overline{\sigma_w}) \quad (42)$$

or also :

$$\bar{\rho} \frac{\partial(\hat{w})}{\partial t} + \bar{\rho} \hat{\vec{u}} \cdot \operatorname{grad}(\hat{w}) = -\operatorname{div}(\overline{\rho w'' \vec{u}''}) - \frac{\partial(\sum_k \overline{\rho_k \tilde{w}_k^2})}{\partial z} - \bar{\rho} g - \frac{\partial \bar{p}}{\partial z} + \operatorname{div}(\overline{\sigma_w}) \quad (43)$$

The second term on the right hand side of this last equation may be decomposed in :

$$\begin{aligned} -\frac{\partial(\sum_k \overline{\rho_k \tilde{w}_k^2})}{\partial z} &= -\frac{\partial(\sum_k \overline{\rho q_k \tilde{w}_k^2})}{\partial z} \\ &= -\frac{\partial(\sum_k \bar{\rho} \hat{q}_k \hat{w}_k^2)}{\partial z} - \frac{\partial(\sum_k \bar{\rho} q_k'' \hat{w}_k^2)}{\partial z} - 2 \frac{\partial(\sum_k \bar{\rho} \hat{q}_k \hat{w}_k \tilde{w}_k'')}{\partial z} \\ &\quad - 2 \frac{\partial(\sum_k \bar{\rho} q_k'' \hat{w}_k \tilde{w}_k'')}{\partial z} - \frac{\partial(\sum_k \bar{\rho} q_k'' (\tilde{w}_k'')^2)}{\partial z} \end{aligned}$$

We will usually keep only the first term of this development. The average NH equation for the vertical velocity is then finally :

$$\bar{\rho} \frac{\partial(\hat{w})}{\partial t} + \bar{\rho} \hat{\vec{u}} \cdot \operatorname{grad}(\hat{w}) = -\operatorname{div}(\overline{\rho w'' \vec{u}''}) - \frac{\partial(\sum_k \bar{\rho} \hat{q}_k \hat{w}_k^2)}{\partial z} - \bar{\rho} g - \frac{\partial \bar{p}}{\partial z} + \operatorname{div}(\overline{\sigma_w}) \quad (44)$$

## 9 The internal energy and the enthalpy of a mixture

### 9.1 Case of a monophasic perfect gas system

Let  $e_i$  be the local internal energy of the monophasic system. If the system is a mixture of perfect gases only,  $e_i = c_v T$ .

The specific enthalpy of a monophasic system is defined as  $h = e_i + p/\rho$ . In the case of a perfect gas,  $h$  is a function of the temperature only :

$$h = h^\circ + c_p T$$

where  $h^\circ$  is a reference enthalpy and  $c_p$  is the massic heat (both  $h^\circ$  and  $c_p$  are supposed constant). Note that

$$\frac{\partial h}{\partial T} = c_p$$

The average specific enthalpy is logically defined by :

$$\hat{h} = \frac{1}{\bar{\rho}V} \int_V \rho h dv$$

Using the detailed expression of  $h$ , we get :

$$\hat{h} = \frac{1}{\bar{\rho}V} \int_V \rho (h^\circ + c_p T) dv = h^\circ + c_p \hat{T}$$

where

$$\hat{T} = \frac{1}{\bar{\rho}V} \int_V \rho T dv$$

## 9.2 Case of a multiphase system

With hypothesis **H1**, we suppose that all the species in the parcel have the same local temperature :  $T_k = T$  for  $k = 0, n$ .

The local internal energy of the mixture is :

$$e_i = \sum_k q_k e_{ik}$$

where  $e_{ik}$  is the specific local internal energy of the species  $k$ .

**H9** : In this paper, we suppose that the liquid and solid phases do not contribute to the local pressure.

In that case, the enthalpy of each species  $l$  may be written as :

$$h_l = e_{il} + p_l / \rho_l$$

where  $p_l = 0$  for the liquid and solid species.

By definition, the local enthalpy of the mixture as the mass-weighted value is :

$$\rho h = \sum_k \rho_k h_k$$

then :

$$h = \frac{1}{\rho} \sum_k [\rho_k e_{ik} + p_k] = e_i + \frac{p}{\rho}$$

**H10** : If we neglect the pressure dependency of the enthalpy of the liquid and solid phases, we can also write the local specific enthalpy of any species  $l$  as :

$$h_l = h_l^\circ + c_{p_l} T$$

where  $h_l^\circ$  is a reference specific enthalpy for the species  $l$  and  $c_{p_l}$  is the massic heat of the species  $l$  (the  $h_l^\circ$  and  $c_{p_l}$  are supposed to be constant).

The local enthalpy of the mixture is also written as :

$$h = \sum_k q_k h_k = \left( \sum_k q_k h_k^\circ \right) + \left( \sum_k q_k c_{p_k} \right) T = h^\circ + c_p T$$

**but**, in this case,  $h^\circ = \sum_k q_k h_k^\circ$  and  $c_p = \sum_k q_k c_{p_k}$  are not constant because the composition of the mixture may vary (the  $q_i$  are pronostic variables).

The average value of the enthalpy of the mixture is then :

$$\widehat{h} = \frac{1}{m} \int_V \rho \left( \sum_k h_k q_k \right) dv = \frac{1}{m} \int_V \rho h dv = \widehat{h}^\circ + \widehat{c}_p \widehat{T}$$

where  $\widehat{h}^\circ = \sum_k \widehat{q}_k h_k^\circ$

Rigorously, this last equation should be written :

$$\widehat{h} = \widehat{h}^\circ + \widehat{c}_p \widehat{T} + \frac{1}{\rho} \overline{\rho c_p'' T''}$$

where  $\widehat{c}_p = \sum_k \widehat{q}_k c_{p_k}$

**H11** : But, in the following, we will neglect the last term on the right hand side in order to conserve the classical thermodynamic relationships :

$$\frac{\partial \widehat{h}}{\partial \widehat{T}} = \widehat{c}_p$$

## 10 Thermodynamic equation

The classical method to deduce the thermodynamic equation is to substract the equation for the kinetic energy from the equation for the total energy (kinetic energy + internal energy). It is much simpler to do this operation at the local scale, and to apply the average operator in a second step, directly on the thermodynamic equation.

### 10.1 Case with hydrostatic approximation

In the hydrostatic case, the contribution of the vertical velocity in the kinetic energy is neglected. Locally, the mixture specific kinetic energy is then :

$$e_c = \frac{1}{2} (u^2 + v^2)$$

(Let's remind that the local values of  $u$  and  $v$  are the same for all the species thanks to the hypothesis of local horizontal mechanic equilibrium).

Note that in the following, the Coriolis force will be simplified as it is usually done when we adopt the « thin shell » approximation ( $z \ll a$ ).

#### 10.1.1 Local kinetic energy equation

The equation for the local value of  $e_c$  is obtained after multiplication of 1.34 by  $u$ , multiplication of 1.35 by  $v$  and addition of the results of these two operations. We get :

$$\rho \frac{De_c}{Dt} = \rho \left[ \frac{\partial (e_c)}{\partial t} + \vec{u} \cdot \vec{\text{grad}}(e_c) \right] = -\vec{u}_h \cdot \vec{\text{grad}}_h p + u \text{div} \sigma_u + v \text{div} \sigma_v \quad (45)$$

#### 10.1.2 Local total energy equation

In the hydrostatic case, the local specific total energy is  $e_t = e_i + e_c = 1/\rho \sum_k (\rho_k e_{ik}) + 1/2(u^2 + v^2)$  (see section 1.8).

The application of the first principle of thermodynamics for any volume  $V$  gives :

$$\frac{\partial \int_V \sum_k \rho_k e_{tk} dv}{\partial t} = \frac{\partial \int_V \rho e_t dv}{\partial t} = - \underbrace{\int_V \operatorname{div} \left( \sum_k \rho_k e_{tk} \vec{u}_k \right) dv}_A + \underbrace{\dot{W}}_B + \underbrace{\dot{Q}}_C$$

where

- term  $A$  is the budget of 3D local transport of total energy of the different species ;
- term  $B$  is the work by time unit (power) of the external forces applied to the volum  $V$  ;
- term  $C$  is the heat exchanged by the volum  $V$  with the environment.

$$\begin{aligned} A &= - \int_V \operatorname{div} \left( \sum_k \rho_k e_{tk} \vec{u}_k \right) dv = - \int_V \operatorname{div} \left( \sum_k \rho_k e_{ik} \vec{u}_k \right) dv - \int_V \operatorname{div} \left( \sum_k \rho_k e_c \vec{u}_k \right) dv \\ &= - \int_V \operatorname{div} \left( \underbrace{\left[ \sum_k \rho_k e_{ik} \right]}_{\rho e_i} \vec{u} \right) dv - \int_V \operatorname{div} \left( \sum_k \rho_k e_{ik} \vec{u}_k \right) dv - \int_V \operatorname{div} \left( \underbrace{\left[ \sum_k \rho_k \vec{u}_k \right]}_{\rho \vec{u}} e_c \right) dv \\ &= - \int_V \operatorname{div} (\rho (e_i + e_c) \vec{u}) dv - \int_V \operatorname{div} \left( \sum_k \rho_k e_{ik} \vec{u}_k \right) dv \\ &= - \int_V \operatorname{div} (\rho e_t \vec{u}) dv - \int_V \frac{\partial (\sum_k \rho_k e_{ik} \vec{u}_k)}{\partial z} dv \end{aligned}$$

The work (by time unit) of the gravity is  $-\rho g w$ . In the hydrostatic case, this term is balanced by the work of the Archimed force :

$$\frac{\partial p}{\partial z} w = -\rho g w \quad (46)$$

The work budget of the pressure forces around the surface  $S$  is :

$$\int_S \sum_k [(-p_k \vec{n}) \cdot \vec{u}_k] ds = \int_V \operatorname{div} \left( \sum_k [-p_k \vec{u}_k] \right) dv \quad (47)$$

where the  $p_k$  are supposed to be zero for the liquid and solid species (but we will keep this complete form which allows an easier expression for the enthalpy equation).

The work of the Coriolis force is zero (the Coriolis force is always orthogonal to the velocity).

The work of the viscous force may be written as :

$$\int_S (\vec{\sigma} \cdot \vec{n}) \cdot \vec{u} ds = \int_V \operatorname{div} (\vec{\sigma} \cdot \vec{u}) dv \quad (48)$$

The heat exchanged at the border  $S$  of the volume  $V$  with the environment is due to conduction and radiation. Let be  $\vec{J}_Q$  the vector of total heat flux density. Then :

$$C = \int_S \vec{J}_Q \cdot \vec{n} ds = \int_V \operatorname{div} (\vec{J}_Q) dv$$

So, finally, the evolution of the local specific total energy is :

$$\begin{aligned} \int_V \left( \frac{\partial \rho e_t}{\partial t} \right) dv &= - \int_V \operatorname{div} (\rho e_t \vec{u}) dv - \int_V \frac{\partial (\sum_k \rho_k e_{ik} \vec{u}_k)}{\partial z} dv \\ &\quad - \int_V \rho g w + \int_V \operatorname{div} \left[ \sum_k (-p_k \vec{u}_k) \right] dv + \int_V \operatorname{div} (\vec{\sigma} \cdot \vec{u}) dv + \int_V \operatorname{div} (\vec{J}_Q) \end{aligned} \quad (49)$$



As this is valid for any volume  $V$ , we can also write :

$$\left( \frac{\partial \rho e_t}{\partial t} \right) = -\text{div}(\rho e_t \vec{u}) - \frac{\partial (\sum_k \rho_k e_{i_k} \tilde{w}_k)}{\partial z} - \rho g w + \text{div} \left[ \sum_k (-p_k \vec{u}_k) \right] + \text{div}(\vec{\sigma} \cdot \vec{u}) + \text{div}(\vec{J}_Q) \quad (50)$$

### 10.1.3 Local thermodynamic equation

Substracting equation 1.45 [ + the vertical contribution 1.46) from equation 1.50 ], we obtain the evolution equation for the local internal energy :

$$\rho \left[ \frac{\partial (e_i)}{\partial t} + \vec{u} \cdot \text{grad}(e_i) \right] = - \frac{\partial (\sum_k (\rho_k e_{i_k} + p_k) \tilde{w}_k)}{\partial z} - p \text{div}(\vec{u}) + \epsilon + \text{div}(\vec{J}_Q) \quad (51)$$

where we note  $\epsilon$  the dissipation of the internal energy by the viscous stress ( $\epsilon = \text{div}(\vec{\sigma} \cdot \vec{u}) - u \text{div} \sigma_u - v \text{div} \sigma_v$ ).

This last equation is easily transformed in an equation for the local enthalpy of the mixture  $h = e_i + \frac{p}{\rho}$  (see section 1.8) :

$$\rho \left[ \frac{\partial (h)}{\partial t} + \vec{u} \cdot \text{grad}(h) \right] = \frac{Dp}{Dt} - \frac{\partial (\sum_k \rho_k h_k \tilde{w}_k)}{\partial z} + \epsilon + \text{div}(\vec{J}_Q) \quad (52)$$

Using the continuity equation, this equation may also be written as :

$$\frac{\partial (\rho h)}{\partial t} + \text{div}(\rho h \vec{u}) = \frac{Dp}{Dt} - \frac{\partial (\sum_k \rho_k h_k \tilde{w}_k)}{\partial z} + \epsilon + \text{div}(\vec{J}_Q) \quad (53)$$

This last equation can be transformed in an equation for the local temperature  $T$  or for the variable  $\rho c_p T$ . With such a manipulation, we show explicitly the phase change terms in the thermodynamics equation.

The specific enthalpy of the mixture is  $h = \sum_k q_k (h^\circ_k + c_{p_k} T) = h^\circ + c_p T$ . Replacing  $h$  by this expression in equation 1.53, we obtain :

$$\frac{\partial (\rho c_p T)}{\partial t} + \text{div}(\rho c_p T \vec{u}) = \frac{Dp}{Dt} - \frac{\partial (\sum_k \rho_k h_k \tilde{w}_k)}{\partial z} + \epsilon + \text{div}(\vec{J}_Q) - \sum_k h^\circ_k \left[ \frac{\partial \rho q_k}{\partial t} + \text{div}(\rho q_k \vec{u}) \right] \quad (54)$$

From equation 1.22, the last term on the right hand side may be transform and the equation becomes :

$$\frac{\partial (\rho c_p T)}{\partial t} + \text{div}(\rho c_p T \vec{u}) = \frac{Dp}{Dt} - \frac{\partial (\sum_k \rho_k h_k \tilde{w}_k)}{\partial z} + \epsilon + \text{div}(\vec{J}_Q) - \sum_k h^\circ_k \left[ - \frac{\partial (\rho_k \tilde{w}_k)}{\partial z} + \dot{\rho}_k \right] \quad (55)$$

But this equation simplifies because the terms in  $h^\circ_k \dot{\rho}_k$  may be combined in a way such as two latent heats at  $T=0$  K appear :

$$\begin{aligned} \frac{\partial (\rho c_p T)}{\partial t} + \text{div}(\rho c_p T \vec{u}) &= \frac{Dp}{Dt} - \underbrace{\frac{\partial (\sum_k \rho_k h^\circ_k \tilde{w}_k)}{\partial z}}_A - \frac{\partial (\sum_k \rho_k c_{p_k} T \tilde{w}_k)}{\partial z} \\ &+ \epsilon + \text{div}(\vec{J}_Q) - \underbrace{\sum_k h^\circ_k \left[ - \frac{\partial (\rho_k \tilde{w}_k)}{\partial z} \right]}_B + L_v(T=0) \dot{\rho}_l + L_i(T=0) \dot{\rho}_i \end{aligned} \quad (56)$$

where  $\dot{\rho}_l$  is the global source/sink of the liquid species and  $\dot{\rho}_i$  is the global source/sink of the solid species.

The combination of term  $A$  and  $B$  of this equation gives :

$$-\frac{\partial(\sum_k \rho_k h_k^\circ \tilde{w}_k)}{\partial z} + \sum_k h_k^\circ \left[ -\frac{\partial(\rho_k \tilde{w}_k)}{\partial z} \right] = -\sum_k \rho_k \tilde{w}_k \underbrace{\frac{\partial h_k^\circ}{\partial z}}_{=0} = 0$$

Then, the final form of the thermodynamic equation is :

$$\frac{\partial(\rho c_p T)}{\partial t} + \text{div}(\rho c_p T \vec{u}) = \frac{Dp}{Dt} \left\{ -\frac{\partial(\sum_k \rho_k c_{pk} T \tilde{w}_k)}{\partial z} \right\} + \epsilon + \text{div}(\vec{J}_Q) + L_v(T=0)\dot{\rho}_l + L_i(T=0)\dot{\rho}_i \quad (57)$$

An other equivalent form of this equation is :

$$\rho c_p \frac{\partial T}{\partial t} + \rho c_p \vec{u} \cdot \text{grad} T = \frac{Dp}{Dt} - \frac{\partial(\sum_k \rho_k h_k \tilde{w}_k)}{\partial z} + \epsilon + \text{div}(\vec{J}_Q) - \sum_k h_k \left[ -\frac{\partial(\rho_k \tilde{w}_k)}{\partial z} + \dot{\rho}_k \right] \quad (58)$$

The  $h_k \dot{\rho}_k$  terms may be combined in a way that two latent heats (we choose here  $L_v$  and  $L_i$ ) appear in this equation (see box ?? for details) :

$$\rho c_p \frac{\partial T}{\partial t} + \rho c_p \vec{u} \cdot \text{grad} T = \frac{Dp}{Dt} \left\{ -\sum_k \left( \begin{array}{c} \rho_k \tilde{w}_k \\ \frac{\partial h_k}{\partial z} \\ \partial(c_{pk} T)/\partial z \end{array} \right) \right\} + \epsilon + \text{div}(\vec{J}_Q) + L_v(T)\dot{\rho}_l + L_i(T)\dot{\rho}_i \quad (59)$$

#### 10.1.4 Averaged thermodynamic equation

We now apply the (volumic) average operator to the local thermodynamic equation. For this operation, we chose the more linear form 1.57 of the thermodynamic equation :

$$\underbrace{\frac{1}{V} \int_V \left[ \frac{\partial(\rho c_p T)}{\partial t} \right] dv}_A + \underbrace{\frac{1}{V} \int_V [\text{div}(\rho c_p T \vec{u})] dv}_B = \underbrace{\frac{1}{V} \int_V \left[ \frac{Dp}{Dt} \right] dv}_C \quad (60)$$

$$- \underbrace{\frac{1}{V} \int_V \left[ \frac{\partial(\sum_k \rho_k c_{pk} T \tilde{w}_k)}{\partial z} \right] dv}_D$$

$$+ \underbrace{\frac{1}{V} \int_V \epsilon dv}_E + \underbrace{\frac{1}{V} \int_V [\text{div}(\vec{J}_Q)] dv}_F$$

$$+ \underbrace{\frac{1}{V} \int_V [L_v(T=0)\dot{\rho}_l + L_i(T=0)\dot{\rho}_i] dv}_G$$

In a first stage, we will express rigorously each of the terms of this equation, and then, we will do some necessary approximation.

**term A :**

$$\frac{1}{V} \int_V \frac{\partial(\rho c_p T)}{\partial t} = \frac{\partial \overline{\rho c_p T}}{\partial t} + \frac{\partial \overline{\rho c_p'' T''}}{\partial t}$$

**term B :**

$$\frac{1}{V} \int_V \text{div}(\rho c_p T \vec{u}) dv = \text{div}[\overline{\rho c_p T \vec{u}} + \overline{\rho c_p'' T'' \vec{u}''} + \overline{\rho c_p'' T'' \vec{u}} + \overline{\rho c_p T \vec{u}''} + \overline{\rho c_p'' T'' \vec{u}''}]$$

**term C :**

$$\frac{1}{V} \int_V \left[ \frac{Dp}{Dt} \right] dv = \underbrace{\frac{\partial \bar{p}}{\partial t} + \widehat{u} \cdot \vec{\text{grad}} \bar{p}}_{\widehat{\frac{D\bar{p}}{Dt}}} + \underbrace{\overline{\frac{\rho' \tilde{u}'' \bar{p}}{\tilde{u}''}} \cdot \vec{\text{grad}} \bar{p} + \overline{\tilde{u}'' \cdot \text{grad} p'}}_{\text{presso-correlation}}$$

**term D :**

$$\frac{1}{V} \int_V \left[ \frac{\partial (\sum_k \rho_k c_{pk} T \tilde{w}_k)}{\partial z} \right] dv = \sum_k c_{pk} \frac{\partial \left[ \overline{\rho \hat{q}_k \hat{T} \hat{w}_k} + \overline{\rho q_k'' T'' \tilde{w}_k''} + \overline{\rho q_k'' T'' \hat{w}_k} + \overline{\hat{q}_k \rho T'' \tilde{w}_k''} + \overline{\hat{T} \rho q_k'' \tilde{w}_k''} \right]}{\partial z}$$

**term E :**

$$\frac{1}{V} \int_V \epsilon dv = \bar{\epsilon}$$

**term F :**

$$\frac{1}{V} \int_V \left[ \text{div}(\vec{J}_Q) \right] dv = \overline{\text{div}(\vec{J}_Q)}$$

**term G :**

$$\frac{1}{V} \int_V [L_v(T=0) \rho_l + L_i(T=0) \rho_i] dv = L_v(T=0) \bar{\rho}_l + L_i(T=0) \bar{\rho}_i$$

Neglecting all the term with a perturbation of  $c_p$  or a perturbation of  $\tilde{w}_k$  and the presso-correlations, we finally get the following averaged thermodynamic equation :

$$\begin{aligned} \frac{\partial \bar{\rho} \widehat{c}_p \widehat{T}}{\partial t} + \text{div}(\widehat{\rho} \widehat{c}_p \widehat{T} \widehat{u}) &= \text{div}(\widehat{c}_p \overline{\rho T'' \tilde{u}''}) + \frac{\partial \bar{p}}{\partial t} + \widehat{u} \cdot \vec{\text{grad}} \bar{p} \\ &\quad - \frac{\partial \left[ \sum_k (\bar{\rho} c_{pk} \widehat{q}_k \widehat{T} \widehat{w}_k) \right]}{\partial z} + \bar{\epsilon} + \overline{\text{div}(\vec{J}_Q)} + L_v(T=0) \bar{\rho}_l + L_i(T=0) \bar{\rho}_i \end{aligned} \quad (61)$$

*Remarque : Cette equation peut très facilement etre transformée pour obtenir l'équation 20 du papier de Bart.*

*Pour cela, on écrit en utilisant les equations pour la concentration des différentes especes que :*

$$\begin{aligned} \frac{\dot{\rho}_l}{\bar{\rho}} &= g \frac{\partial}{\partial p} (P_l' - P_l''') \\ \frac{\dot{\rho}_i}{\bar{\rho}} &= g \frac{\partial}{\partial p} (P_i' - P_i''') \end{aligned}$$

*On utilise ensuite le fait que :*

$$\begin{aligned} P_l &= -\bar{\rho}_r \widehat{w}_r \\ P_i &= -\bar{\rho}_s \widehat{w}_s \end{aligned}$$

*et que, avec les approximations déjà mentionnées (on néglige les correlations qui font intervenir les  $\tilde{w}_k$ ) :*

$$\sum_k \bar{\rho}_k \widehat{w}_k = 0$$

*Donc la vitesse des espèces non precipitantes (qu'on suppose identiques pour toute les espèces non precipitantes) est :*

$$\widehat{w}_{np} = \frac{(P_l + P_i)}{\sum_{np} \bar{\rho}_k} = \frac{(P_l + P_i)}{\rho - \bar{\rho}_r - \bar{\rho}_s}$$

En reportant tous ces résultats dans l'équation 1.61, on retrouve l'équation 20 de Bart.

The equation 1.61 can be re-arranged in order to obtain the averaged form corresponding to equation 1.59. The term on the left hand side of equation 1.61 may also be written as :

$$\begin{aligned}
\frac{\partial \overline{\rho \hat{c}_p \hat{T}}}{\partial t} + \text{div}(\overline{\rho \hat{c}_p \hat{T} \hat{u}}) &= \overline{\rho} \frac{\hat{D} \hat{c}_p \hat{T}}{Dt} \\
&= \overline{\rho \hat{c}_p} \frac{\hat{D} \hat{T}}{Dt} + \overline{\rho \hat{T}} \frac{\hat{D} \hat{c}_p}{Dt} \\
&= \overline{\rho \hat{c}_p} \frac{\hat{D} \hat{T}}{Dt} + \overline{\rho \hat{T}} \frac{\hat{D} (\sum_k \hat{q}_k c_{pk})}{Dt} \\
&= \overline{\rho \hat{c}_p} \frac{\hat{D} \hat{T}}{Dt} + \sum_k \left[ \overline{\rho c_{pk} \hat{T}} \frac{\hat{D} \hat{q}_k}{Dt} \right]
\end{aligned}$$

Using equation 1.28, the second term of this expression becomes :

$$\sum_k \left[ \overline{\rho c_{pk} \hat{T}} \frac{\hat{D} \hat{q}_k}{Dt} \right] = \sum_k \left[ c_{pk} \hat{T} \left( -\frac{\partial (\overline{\rho \hat{q}_l \hat{w}_l})}{\partial z} - \frac{\partial (\overline{\rho q_l'' w_l''})}{\partial z} + \overline{\rho_l} \right) \right] \quad (62)$$

The fourth term on the right hand side of equation 1.61 can be splitted in two parts :

$$-\frac{\partial \left[ \sum_k (\overline{\rho c_{pk} \hat{q}_k \hat{T} \hat{w}_k}) \right]}{\partial z} = -\sum_k \left( \overline{\rho_k \hat{w}_k} \frac{\partial c_{pk} \hat{T}}{\partial z} \right) - \sum_k \left( c_{pk} \hat{T} \frac{\partial \overline{\rho_k \hat{w}_k}}{\partial z} \right)$$

After simplification, an other form of equation 1.61 is :

$$\overline{\rho \hat{c}_p} \frac{\hat{D} \hat{T}}{Dt} = \text{div}(\overline{\hat{c}_p \rho T'' \hat{u}''}) + \frac{\hat{D} \overline{p}}{Dt} + \overline{\epsilon} + \text{div}(\overline{J_Q}) - \sum_k \left( \overline{\rho_k \hat{w}_k} \frac{\partial c_{pk} \hat{T}}{\partial z} \right) + L_v(T) \overline{\rho_l} + L_i(T) \overline{\rho_i} \quad (63)$$

## 10.2 Non hydrostatic case

In the non hydrostatic case, the contribution of the vertical velocity has to be taken into account in the kinetic energy. Locally, the mixture specific « barycentric » kinetic energy is then :

$$e_c = \frac{1}{2} (u^2 + v^2 + w^2)$$

(Let's remind that the local values of  $u$  and  $v$  are the same for all the species thanks to the hypothesis of local horizontal equilibrium).

But, be careful :

$$\sum_k \rho_k e_{ck} = \rho e_c + \sum_k \left( \frac{\rho_k}{2} \tilde{w}_k^2 \right)$$

The residual term on the right hand side is not zero when there is precipitating species. In the following, we will use the notation :

$$\rho \tilde{e}_c = \sum_k \left( \frac{\rho_k}{2} \tilde{w}_k^2 \right)$$

Note that in the following, the Coriolis force will be simplified as it is usually done when we adopt the « thin shell » approximation ( $z \ll a$ ).

### 10.2.1 The local form of the NH « barycentric » kinetic energy equation

The equation of the local « barycentric » kinetic energy is obtain after multiplication of 1.34 by  $u$ , multiplication of 1.35 by  $v$ , multiplication of 1.40 by  $w$  and addition of the results of these three operations. We get :

$$\rho \frac{De_c}{Dt} = \rho \left[ \frac{\partial(e_c)}{\partial t} + \vec{u} \cdot \vec{\text{grad}}(e_c) \right] = -w \frac{\partial(\sum_k \rho_k \tilde{w}_k^2)}{\partial z} - \vec{u} \cdot \vec{\text{grad}} p - \rho g w + u \text{div} \sigma_u + v \text{div} \sigma_v + w \text{div} \sigma_w \quad (64)$$

### 10.2.2 The local form of the NH total energy equation

In the non hydrostatic case, the local « barycentric » specific total energy is  $e_t = e_i + e_c = 1/\rho(\sum_k e_{ik}) + 1/2(u^2 + v^2 + w^2)$ . But the local specific total energy is

$$\frac{1}{\rho} \sum_k \rho_k e_{tk} = e_t + \tilde{e}_c$$

As in the hydrostatic case, the application of the first principle of thermodynamics for any volume  $V$  gives :

$$\frac{\partial \int_V \sum_k \rho_k e_{tk} dv}{\partial t} = \frac{\partial \int_V \rho(e_t + \tilde{e}_c) dv}{\partial t} = - \underbrace{\int_V \text{div} \left( \sum_k \rho_k e_{tk} \vec{u}_k \right) dv}_A + \underbrace{\dot{W}}_B + \underbrace{\dot{Q}}_C$$

In the right hand side, only the term  $A$  shows a difference when compared with the hydrostatic equation. Actually :

$$\begin{aligned} A &= - \int_V \text{div} \left( \sum_k \rho_k e_{tk} \vec{u}_k \right) dv = - \int_V \text{div} \left( \sum_k \rho_k e_{ik} \vec{u}_k \right) dv - \int_V \text{div} \left( \sum_k \rho_k e_{ck} \vec{u}_k \right) dv \\ &= - \int_V \text{div} \left( \underbrace{\left[ \sum_k \rho_k e_{ik} \right]}_{\rho e_i} \vec{u} \right) dv - \int_V \text{div} \left( \sum_k \rho_k e_{ik} \tilde{\vec{u}}_k \right) dv - \int_V \text{div} \left( \underbrace{\left[ \sum_k \rho_k e_{ck} \right]}_{\rho(e_c + \tilde{e}_c)} \vec{u} \right) dv - \int_V \text{div} \left( \sum_k \rho_k e_{ck} \tilde{\vec{u}}_k \right) dv \\ &= - \int_V \text{div}(\rho(e_i + e_c) \vec{u}) dv - \int_V \text{div}(\rho \tilde{e}_c \vec{u}) dv - \int_V \text{div} \left( \sum_k \rho_k (e_{ik} + e_{ck}) \tilde{\vec{u}}_k \right) dv \\ &= - \int_V \text{div}(\rho e_t \vec{u}) dv - \int_V \text{div}(\rho \tilde{e}_c \vec{u}) dv - \int_V \frac{\partial(\sum_k \rho_k (e_{ik} + e_{ck}) \tilde{w}_k)}{\partial z} dv \end{aligned}$$

The local equation for the local total energy becomes :

$$\left( \frac{\partial \rho(e_t + \tilde{e}_c)}{\partial t} \right) = -\text{div}(\rho(e_t + \tilde{e}_c) \vec{u}) - \frac{\partial(\sum_k \rho_k (e_{ik} + e_{ck}) \tilde{w}_k)}{\partial z} - \rho g w - \text{div} \left( \sum_k (-p_k \vec{u}_k) \right) + \text{div}(\vec{\sigma} \cdot \vec{u}) + \text{div}(\vec{J}_Q) \quad (65)$$

### 10.2.3 Local NH thermodynamic equation

Substracting equation 1.64 from equation 1.65, we obtain the evolution equation for the local internal energy + barycentric departure kinetic energy :

$$\begin{aligned} \rho \left[ \frac{\partial(e_i + \tilde{e}_c)}{\partial t} + \vec{u} \cdot \vec{\text{grad}}(e_i + \tilde{e}_c) \right] &= - \frac{\partial(\sum_k (\rho_k e_{ik} + p_k) \tilde{w}_k)}{\partial z} \\ &\quad \left[ - \frac{\partial(\sum_k (\rho_k e_{ck}) \tilde{w}_k)}{\partial z} - w \frac{\partial(\sum_k \rho_k \tilde{w}_k^2)}{\partial z} \right] \\ &\quad - p \text{div}(\vec{u}) + \epsilon + \text{div}(\vec{J}_Q) \end{aligned} \quad (66)$$

where we note  $\epsilon$  the dissipation of the internal energy by the viscous stress ( $\epsilon = \text{div}(\vec{\sigma} \cdot \vec{u}) - u \text{div} \sigma_u - v \text{div} \sigma_v - w \text{div} \sigma_w$ ). The terms in brackets may be combined :

$$\begin{aligned}
\left[ -\frac{\partial (\sum_k (\rho_k e c_k) \tilde{w}_k)}{\partial z} - w \frac{\partial (\sum_k \rho_k \tilde{w}_k^2)}{\partial z} \right] &= -\frac{\partial (\sum_k (\frac{\rho_k}{2} (w^2 + \tilde{w}_k^2 + 2w\tilde{w}_k)) \tilde{w}_k)}{\partial z} - w \frac{\partial (\sum_k \rho_k \tilde{w}_k^2)}{\partial z} \\
&= -\frac{\partial (\sum_k (\frac{\rho_k}{2} (\tilde{w}_k^2 + 2w\tilde{w}_k)) \tilde{w}_k)}{\partial z} - w \frac{\partial (\sum_k \rho_k \tilde{w}_k^2)}{\partial z} \\
&= -\frac{\partial (\sum_k (\frac{\rho_k}{2} \tilde{w}_k^3))}{\partial z} - \frac{\partial(w)}{\partial z} [\sum_k \rho_k \tilde{w}_k^2] \\
&= -\frac{\partial (\sum_k (\frac{\rho_k}{2} \tilde{w}_k^3))}{\partial z} - \rho \tilde{e}_c \frac{\partial(w)}{\partial z}
\end{aligned}$$

In practice, we will (at least) neglect the triple correlations of  $\tilde{w}_k$ , but we may keep in the thermodynamic equation the last term on the right hand side (?).

*Pour la suite, je ne sais pas trop quoi faire du terme d'évolution de  $\tilde{e}_c$ . Peut être qu'à ce moment là il faut faire intervenir des hypothèses faites sur les vitesses de chute, genre stationnarité ?*

## 11 Reference averaged equation sets

In this sets of equations, we are doing the « thin shell » approximation.

### 11.1 The hydrostatic case

$$\bar{p} = \bar{\rho}(R_0 \hat{q}_0 + R_1 \hat{q}_1) \hat{T} \quad (67)$$

$$\frac{\hat{D}\bar{p}}{Dt} = -\bar{\rho} \text{div}(\vec{u}) \quad (68)$$

$$\frac{\hat{D}\hat{q}_l}{\bar{\rho} Dt} = -\frac{\partial (\bar{\rho} \hat{q}_l \hat{w}_l)}{\partial z} - \frac{\partial (\bar{\rho} q_l'' w_l''')}{\partial z} + \bar{\rho}_l \quad (69)$$

$$\frac{\hat{D}\hat{u}}{\bar{\rho} Dt} = -\text{div}(\bar{\rho} \vec{u}'' u'') + \bar{\rho} f \hat{v} - \frac{\partial \bar{p}}{\partial x} + \text{div} \bar{\sigma}_u \quad (70)$$

$$\frac{\hat{D}\hat{v}}{\bar{\rho} Dt} = -\text{div}(\bar{\rho} \vec{u}'' v'') - \bar{\rho} f \hat{u} - \frac{\partial \bar{p}}{\partial y} + \text{div} \bar{\sigma}_v \quad (71)$$

$$\frac{\partial \bar{p}}{\partial z} = -\bar{\rho} g \quad (72)$$

$$\begin{aligned}
\frac{\hat{D}\hat{c}_p \hat{T}}{\bar{\rho} Dt} &= \text{div}(\hat{c}_p \overline{\rho T'' u''}) + \frac{\partial \bar{p}}{\partial t} + \vec{u} \cdot \text{grad} \bar{p} \\
&\quad - \frac{\partial \left[ \sum_k (\bar{\rho} c_{p_k} \hat{q}_k \hat{T} \hat{w}_k^2) \right]}{\partial z} + \bar{\epsilon} + \text{div}(\overline{\vec{J}_Q}) + L_v(T=0) \bar{\rho}_l + L_i(T=0) \bar{\rho}_i
\end{aligned} \quad (73)$$

### 11.2 The non hydrostatic case

In the NH case, both the « vertical momentum » equation 1.72 and the thermodynamic equation 1.73 have a different expression :

$$\frac{\hat{D}\hat{w}}{\bar{\rho} Dt} = -\text{div}(\bar{\rho} w'' \vec{u}'') - \frac{\partial (\sum_k \bar{\rho} \hat{q}_k \hat{w}_k^2)}{\partial z} - \bar{\rho} g - \frac{\partial \bar{p}}{\partial z} + \text{div}(\bar{\sigma}_w) \quad (74)$$

$$\begin{aligned}
\rho \left[ \frac{\partial (e_i + \tilde{e}_c)}{\partial t} + \vec{u} \cdot \text{grad}(e_i + \tilde{e}_c) \right] &= - \frac{\partial (\sum_k (\rho_k e_{i_k} + p_k) \tilde{w}_k)}{\partial z} \\
&- \rho \tilde{e}_c \frac{\partial (w)}{\partial z} \\
&- p \text{div}(\vec{u}) + \epsilon + \text{div}(\vec{J}_Q)
\end{aligned} \tag{75}$$