Consitency check with the Arpège/Aladin hydrostatic equations

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1 Introduction

Initially, the hydrostatic equations used in the NWP model Arpège/Aladin where designed for a synoptic scale forcasting system. In this context, the description of the liquid and solid species were based on the pseudo-adiabatic concept : the condensed water species (liquid or solid) are « instantaneously » evacuated from their original air parcel (no cloud or precipitation species in suspension in the atmosphere). A condensation process is then a sink of water vapor and an evaporation process is a source of vapor but the concentrations of liquid and solid water in the atmosphere remain zero.

In practice, in a NWP model, instaneous transformations are replaced by transformations during one time step, such a time step being in Arpège of the order of 15 minutes. During this time step, the liquid and solid phase are falling down to the ground, but they can also evaporate if they cross non saturated layers in the colomn of atmosphere below their level of formation.

The difficulty of the hypothesis about the precipitation done in Arpege/Aladin is that the precipitations never are an ingredient of the air in a parcel because they are always instantaneously evacuated, but these precipitations interacts with the moist air (dry air + water vapor) which is the only ingredient of the atmosphere in this « pseudo-adiabatic » context through transports (fluxes) in and out of the air parcels : if the vertical budget (flux divergence) of the precipitation at a given level is not zero, the precipitations deposit or remove mass, momentum and energy into or from the moist air present at this level.

2 Definition of δm

In the original Arpège/Aladin equations, the application of the pseudo-adiabatic hypothesis is applied with an optionnal refinement in the treatment of the mass fluxes. The two options are known as $\delta m = 0$ or $\delta m = 1$[1].

In the case $\delta m = 1$, if liquid and solid species form they « instantaneously » fall down. Let's suppose that the air is at rest with respect to the ground. In such case, the barycenter of the mass of air originally contained in the (fixed) volume V where the condensation occured is moving down.

In the case $\delta m = 0$, if liquid and solid species form they « instantaneously » fall down but a compensating flux of dry air appears in the volume V.

Note that the case $\delta m = 1$ is more similar with the case considered in the reference equations (Malardel, Geleyn and Bénard, 2005, Part 1, to be found on the Aladin web site).

In any case, the air in any volume of atmosphere never contains hydrometeores, the only constituant are dry air and water vapor. This hypothesis may nevertheless be suspended, and the system could be generalized to a quasi-pseudo-adiabatic case where only non precipitating water remains in suspension.

As all the precipitating species are instantaneously evacuated, $\rho_p = 0$ exept when it is multiplied by the velocity of the precipitation which may be in that case considered as infinite. The product

[1] From cycle 29 of Arpege/Aladin, the definition of δm has changed. This article is dealing only with the original definition of $\delta m = 0$ or $\delta m = 1$ (for the time beiing at least).

of the mass of precipitation by its velocity is the flux of precipitation(in the following, we suppose that we have only one type of precipitation, and we use the subscript p for this precipitation) :

$$P = \rho_p V_p$$

where V_p is the fall velocity of the precipitation (positive value).

A consequence of these hypothesis is :

$$\begin{aligned}
\frac{\partial \rho_p}{\partial t} &= 0 && (1)\\
&= -\text{div}(\rho_p \vec{u}_p) + \dot{\rho}_p && (2)\\
&= -\text{div}(\underbrace{\rho_p}_{=0} \vec{v}) - \text{div}(\underbrace{\rho_p V_p}_{=-P} \vec{k}) + \dot{\rho}_p && (3)
\end{aligned}$$

So finally :

$$\dot{\rho}_p = -\frac{\partial P}{\partial z}$$

where $\dot{\rho}_p$ is the rate of formation/evaporation of liquid (or solid) species.

3 Mass and species budgets in Arpège/Aladin hydrostatic equations

3.1 $\delta m = 1$ case

As we have done from the beginning of this paper, we will examined budgets of mass, momentum and energy in a fixed volume V.

The two "pronostic" componants of the system are the dry air and the vapor. The budget equations for this two constituants are :

$$\begin{aligned}
\frac{\partial \rho_a}{\partial t} &= -\text{div}(\rho_a \vec{u}_a) && (4)\\
\frac{\partial \rho_v}{\partial t} &= -\text{div}(\rho_v \vec{u}_v) + \dot{\rho}_v && (5)
\end{aligned}$$

With our simple hypothesis on the phase changes, we have :

$$\dot{\rho}_v = -\dot{\rho}_p = \frac{\partial P}{\partial z}$$

We also suppose that $\vec{u}_a = \vec{u}_v = \vec{u}$.

The density of the system is $\rho = \rho_a + \rho_v$.

The mass budget of the system is obtained when summing the equation for each componant of the system :

$$\frac{\partial(\rho_a + \rho_v)}{\partial t} = \frac{\partial \rho}{\partial t} = -\text{div}((\rho_a + \rho_v)\vec{u}) + \dot{\rho}_v = -\text{div}(\rho\vec{u}) + \frac{\partial P}{\partial z} \quad (6)$$

In the case $\delta m = 1$, the continuity equation is :

$$\frac{\partial \rho}{\partial t} = -\text{div}(\rho\vec{u}) + \frac{\partial P}{\partial z} \quad (7)$$

or :

$$\frac{D\rho}{Dt} = -\rho\text{div}(\vec{u}) + \frac{\partial P}{\partial z} \quad (8)$$

where $D\ /Dt = \partial\ /\partial t\ + \vec{u}.\vec{\text{grad}}$.

3.2 $\delta m = 0$ case

In the case $\delta m = 0$, we have a dry air source/sink $\dot{\rho}_a$ which compensate the vapor sink/source :

$$\dot{\rho}_a = -\dot{\rho}_v = -\frac{\partial P}{\partial z}$$

The equation for the different species are then written as :

$$\frac{\partial \rho_a}{\partial t} = -\mathrm{div}(\rho_a \vec{u}_a) - \frac{\partial P}{\partial z} \quad (9)$$

$$\frac{\partial \rho_v}{\partial t} = -\mathrm{div}(\rho_v \vec{u}_v) + \frac{\partial P}{\partial z} \quad (10)$$

The continuity equation in the case $\delta m = 0$ is then :

$$\frac{\partial \rho}{\partial t} = -\mathrm{div}(\rho \vec{u}) \quad (11)$$

or

$$\frac{D\rho}{Dt} = -\rho \mathrm{div}(\vec{u}) \quad (12)$$

4 General budget equation with the Arpège/Aladin original hypothesis

Let ψ be any specific variable (variable by mass unit).

4.1 $\delta m = 1$ case

In the case $\delta m = 1$, the budget of ψ in a geometric volume V is given by :

$$\underbrace{\frac{\partial[\rho_a\psi_a + \rho_v\psi_v]}{\partial t}}_{A} = \underbrace{-\mathrm{div}[(\rho_a\psi_a + \rho_v\psi_v)\vec{u_{gaz}}]}_{B} \underbrace{-\mathrm{div}[(\rho_p\psi_p)\vec{u_p}]}_{C} + \underbrace{\dot{S}_\psi}_{D}$$

where

A is the local evolution of the quantity of ψ of the moist air ;

B is the transport budget of the quantity of ψ of the moist air by the velocity of the moist air ;

C is the transport budget of the quantity of ψ of the precipitations by the velocity of the precipitations ;

D is the source of ψ in the moist air.

4.2 $\delta m = 0$ case

In the case $\delta m = 0$, the budget of ψ in a geometric volume V is given by :

$$\frac{\partial[\rho_a\psi_a + \rho_v\psi_v]}{\partial t} = -\mathrm{div}[(\rho_a\psi_a + \rho_v\psi_v)\vec{u_{gaz}}] - \mathrm{div}[(\rho_p\psi_p)\vec{u_p}] \underbrace{+\mathrm{div}[(\rho_p\psi_a)\vec{u_p}]}_{E} + \dot{S}_\psi$$

The term E is the transport budget of the quantity of ψ of the compensating flux of moist air by the the compensating flux of moist air.

4.3 Summary

By definition and hypothesis,

$$\rho_p \vec{u_p} = \underbrace{\rho_p \vec{v}}_{=0} + \underbrace{\rho_p w_p}_{=P} \vec{k} = -P\vec{k}$$

The general budget equation in Arpege/Aladin is then :

$$\frac{\partial[\rho_a \psi_a + \rho_v \psi_v]}{\partial t} = -\text{div}[(\rho_a \psi_a + \rho_v \psi_v)\vec{u_{gaz}}] + \frac{\partial(P\psi_p)}{\partial z} - (1-\delta m)\frac{\partial(P\psi_a)}{\partial z} + \dot{S}_\psi$$

In the case $\delta m = 0$ and if $\psi_p = \psi_a$, the terms C and E balance each other. This is the case for the mass and the horizontal momentum budget, but not for the energy (because the specific heats of water and dry air are different).

5 Horizontal momentum budget in Arpège/Aladin hydrostatic equations

As we have done for the reference équations (Malardel, Geleyn and Bénard, 2005, Part 1), we suppose that the horizontal velocity is the same for all the species.

5.1 $\delta m = 1$ case

The horizontal momentum budget is simply written as :

$$\frac{\partial \rho \vec{v}}{\partial t} = -\text{div}(\rho \vec{v}\vec{u}) + \frac{\partial P\vec{v}}{\partial z} - \rho f\vec{k} \wedge \vec{v} - \vec{\text{grad}}_\text{h}(p) \tag{13}$$

with $\rho = \rho_a + \rho_v$.

$$\rho \left[\frac{\partial \vec{v}}{\partial t} + \vec{u}.\vec{\text{grad}}(\vec{v})\right] = -\vec{v}\frac{\partial \rho}{\partial t} - \vec{v}\text{div}(\rho\vec{u}) + \frac{\partial P\vec{v}}{\partial z} - \rho f\vec{k} \wedge \vec{v} - \vec{\text{grad}}_\text{h}(p)$$

Using the continuity equation in the case $\delta m = 1$, the « Lagrangian » form of this equation is :

$$\rho\frac{D\vec{v}}{Dt} = -\frac{\partial P}{\partial z}\vec{v} + \frac{\partial P\vec{v}}{\partial z} - \rho f\vec{k} \wedge \vec{v} - \vec{\text{grad}}_\text{h}(p)$$

The first term on the right hand side is a sink/source of momentum linked with the formation/evaporation of the precipitation (if the volume V is loosing water because of precipitation, it looses also the momentum associated with this mass of water. In this equation, we suppose implicitly that the precipitation which are formed in a layer conserve the horizontal speed of that layer. But we also suppose that the water vapor wich is evaporated from some precipitation also get the horizontal speed of that layer).

The second term on the right hand side is the budget of the horizontal momentum put in and taken out of the parcel by the precipitations.

Combining the two first terms on the right hand side gives finally the following equation :

$$\rho\frac{D\vec{v}}{Dt} = P\frac{\partial \vec{v}}{\partial z} - \rho f\vec{k} \wedge \vec{v} - \vec{\text{grad}}_\text{h}(p)$$

5.2 $\delta m = 0$ case

In the case $\delta m = 0$, the horizontal momentum equation is even more simple :

$$\frac{\partial \rho \vec{v}}{\partial t} = -\text{div}(\rho \vec{v}\vec{u}) + \frac{\partial P\vec{v}}{\partial z} - \frac{\partial P\vec{v}}{\partial z} - \rho f\vec{k} \wedge \vec{v} - \vec{\text{grad}}_\text{h}(p) \tag{14}$$

or, using the form of the continuity equation in the case $\delta m = 0$:

$$\rho \frac{D\vec{v}}{Dt} = -\rho f \vec{k} \wedge \vec{v} - \vec{\mathrm{grad}}_{\mathrm{h}}(p)$$

The sink/source of momentum link with the precipitation in a layer is compensated by an opposite flux of dry air.

6 The hydrostatic relationship in Arpège/Aladin hydrostatic equations

When we suppose that there is a (quasi) vertical equilibrium between the gravity and the vertical componant of the pressure force in a system without liquid or solid phase in suspension, we get in both cases $\delta m = 1$ and $\delta m = 0$:

$$0 = -\rho g - \frac{\partial p}{\partial z}$$

where $\rho = \rho_a + \rho_v$.

7 The energy budget in Arpège/Aladin hydrostatic equations

7.1 $\delta m = 1$ case

The energy budget of the system of moist air can be written as :

$$\frac{\partial[\rho_a(e_i+e_c)_a + \rho_v(e_i+e_c)_v]}{\partial t} = -\mathrm{div}[\rho_a(e_i+e_c)_a\vec{u}_a + \rho_v(e_i+e_c)_v\vec{u}_v]) + \frac{\partial(Pe_{ip} + Pe_{cp})}{\partial z} + \dot{W} + \dot{Q}$$

with $e_{ca} = e_{cv} = 1/2\vec{v}^2$.

The work (by unit of time) of the exterior forces applied to the system « moist air » is :

$$\dot{W} = -\mathrm{div}(p\vec{u}) + \rho\vec{g}.\vec{u} = -p\mathrm{div}(\vec{u}) - \vec{u}.\vec{\mathrm{grad}}(p) - \rho g w$$

From the equation for the momentum, we deduce the equation for the kinetic energy e_c :

$$\frac{\partial(\rho e_c)}{\partial t} = -\mathrm{div}(\rho e_c) - \vec{v}.\vec{\mathrm{grad}}_{\mathrm{h}} p + \vec{v}.\frac{\partial(P\vec{v})}{\partial z}$$

The last term of this equation is the budget of the kinetic energy fluxes of the precipitation $\partial(Pe_c)/\partial z$.

Substracting the equation for the kinetic energy from the equation for the total energy, we get (the flux of kinetic energy of the precipitation is balanced by the flux of kinetic energy coming from the kinetic energy equation present only in the case $\delta m = 1$) :

$$\frac{\partial(\rho e_i)}{\partial t} = -\mathrm{div}(\rho e_i) + \frac{\partial(Pe_{ip})}{\partial z} - p\mathrm{div}(\vec{u}) + \mathrm{div}(\vec{J}_Q) \tag{15}$$

Using the classical form of the enthalpy for perfect gaz $h_a = c_{p_a}T$ and $h_v = c_{p_v}T$ and neglecting the pressure in the expression of the enthalpy of the precipitating phase $h_p = e_{ip}$, the former equation is transformed into an equation for the enthalpy $h = q_a c_{p_a} T + q_v c_{p_v} T$ of the system (box 1). We also use the notation $c_{p_h} = q_a c_{p_a} + q_v c_{p_v}$.

$$\rho\frac{D(c_{p_h}T)}{Dt} = \frac{Dp}{Dt} - \frac{\partial\left(P(h-h_p)\right)}{\partial z} - P\frac{\partial h}{\partial z} + \mathrm{div}(\vec{J}_Q)$$

This equation may also be written as :

$$\rho\frac{D(c_{p_h}T)}{Dt} = \frac{Dp}{Dt} - \frac{\partial\left(P(L_v(T) + (1-q_v)(c_{p_a} - c_{p_v})T)\right)}{\partial z} - P\frac{\partial(c_{p_h}T)}{\partial z} + \mathrm{div}(\vec{J}_Q)$$

Moving the first term of the right hand side of equation 15 to the left hand side, we get :

$$\rho\frac{D(e_i)}{Dt} + e_i \underbrace{\left(\frac{\partial\rho}{\partial t} + \mathrm{div}(\rho\vec{u})\right)}_{\frac{\partial P}{\partial z}} = \frac{\partial(Pe_{ip})}{\partial z} - p\left(-\frac{1}{\rho}\frac{D\rho}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial z}\right) + \mathrm{div}(\vec{J}_Q) \tag{16}$$

or also

$$\rho\frac{D(e_i)}{Dt} = \frac{Dp}{Dt} - \rho\frac{Dp/\rho}{Dt} - e_i\frac{\partial P}{\partial z} - \frac{p}{\rho}\frac{\partial p}{\partial z} + \frac{\partial(Pe_{ip})}{\partial z} + \mathrm{div}(\vec{J}_Q) \tag{17}$$

and then :

$$\rho\frac{D(h)}{Dt} = \frac{Dp}{Dt} - h\frac{\partial P}{\partial z} + \frac{\partial(Ph_p)}{\partial z} + \mathrm{div}(\vec{J}_Q) \tag{18}$$

Combining the second and the third term of this equation, we finally have :

$$\rho\frac{D(h)}{Dt} = \frac{Dp}{Dt} - \frac{\partial\left(P(h-h_p)\right)}{\partial z} - P\frac{\partial h}{\partial z} + \mathrm{div}(\vec{J}_Q) \tag{19}$$

Box 1:

7.2 $\delta m = 0$ case

In the case $\delta m = 0$, the energy budget of the system is written as :

$$\frac{\partial[\rho_a(e_i+e_c)_a + \rho_v(e_i+e_c)_v]}{\partial t} = -\mathrm{div}[\rho_a(e_i+e_c)_a\vec{u}_a + \rho_v(e_i+e_c)_v\vec{u}_v]) + \frac{\partial(Pe_{ip}+Pe_{cp})}{\partial z} - \frac{\partial(Pe_{ia}+Pe_{ca})}{\partial z} + \dot{W} + \dot{Q}$$

The term $\dot{W}$ is the same than in $\delta m = 1$:

$$\dot{W} = -p\mathrm{div}(\vec{u}) - \vec{u}.\vec{\mathrm{grad}}(p) - \rho g w$$

The kinetic energy equation is :

$$\frac{\partial(\rho e_c)}{\partial t} = -\mathrm{div}(\rho e_c) - \vec{v}.\vec{\mathrm{grad}}_{\mathrm{h}}p$$

Then, after substraction, we get (the flux of kinetic of dry air balances the flux of kinetic of the precipitations, but this is not true for the internal energy) :

$$\frac{\partial(\rho e_i)}{\partial t} = -\mathrm{div}(\rho e_i) + \frac{\partial(Pe_{ip})}{\partial z} - \frac{\partial(Pe_{ia})}{\partial z} - p\mathrm{div}(\vec{u}) + \mathrm{div}(\vec{J}_Q)$$

which may also be transformed in an enthalpy equation :

$$\rho\frac{D(c_{p_h}T)}{Dt} = \frac{Dp}{Dt} + \frac{\partial\left(P(hp - e_{ia})\right)}{\partial z} + \mathrm{div}(\vec{J}_Q)$$

And finally :

$$\rho\frac{D(c_{p_h}T)}{Dt} = \frac{Dp}{Dt} - \frac{\partial\left(P(L_v(T) - c_{p_v}T + c_{v_a}T)\right)}{\partial z} + \mathrm{div}(\vec{J}_Q)$$

8 Discussion

- This system of equations is valid only if there is no liquid or solid water in suspension. It can be generalized to some kind of quasi « pseudo adiabatic » case where non precipitating water remains in suspension in the atmosphere but the precipitating water is instantaneously evacuated (à faire).

– It's possible to go from the complete system of reference equation (Malardel, Geleyn and Bénard, 2005, Part 1) to the pseudo-adiabatic system deduced in chapter 1 **only** in the case $\delta m = 1$. The pseuso-adiabatic solution is then the assymptotic solution when the density of the condensed phases of water goes (tends ?) to zero except in the flux divergence terms where the precipitation density is multiplied by the fall velocity of the precipitation (zero $\times$ infinity $= P$).