A semi-implicit NH quasi-elastic soundproof mass-based dynamical system for NWP applications

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1. Elaboration of a sound-proof NWP system free from the numerical adverse effects of the fast vertically propagation acoustic waves.
2. Maintain the existing AROME/ALARO Dynamical software.

Basic underlying idea :

The actual density $\rho(\theta, p)$ is replaced by a pseudo-density $\widetilde{\rho}(\theta) \Rightarrow$ approximate mass-continuity equation:

$$\frac{D\widetilde{\rho}}{Dt} = \left(\frac{\partial\widetilde{\rho}}{\partial\theta}\right)_p \frac{D\theta}{Dt} + \underbrace{\left(\frac{\partial\widetilde{\rho}}{\partial p}\right)_\theta}_{=0} \frac{Dp}{Dt} = -\widetilde{\rho}\, D_3$$

1. Quasi-elastic approximation: $\widetilde{\rho} = \widetilde{\rho}[\theta(x, y, z, t)] \Rightarrow$ the most accurate sound-proof approx. with no evidence of Rossby waves distortion *[Arakawa and Konor, 2009]* $\Rightarrow$ viable at synoptic and meso-scales .

Basic definitions and approximations

QE reference state

$(\pi, \widetilde{\rho}, \widetilde{T})$ are thermodynamic variables of the QE reference state defined as

$$\begin{aligned} \widetilde{\rho} &= p_{00}^{\kappa} \frac{\pi^{1-\kappa}}{R\theta} \equiv \widetilde{\rho}(\pi, \theta) \\ \frac{\partial \pi}{\partial z} &= -\widetilde{\rho}\, g \\ \widetilde{T} &= \frac{\pi}{R\widetilde{\rho}} \end{aligned}$$

Hence, $\pi = \pi(\theta) \Rightarrow \widetilde{\rho} = \widetilde{\rho}(\theta)$
Defining pressure departure as $q = \ln(p/\pi) \Rightarrow$

$$\begin{aligned} p &= \pi \exp[q] \\ \rho &= \widetilde{\rho} \exp[(1-\kappa) q] \\ T &= \widetilde{T} \exp[\kappa\, q] \end{aligned}$$

QE approximation

$$\frac{\dot{\widetilde{\rho}}}{\widetilde{\rho}} = -D_3 \qquad \textit{(QE mass-continuity eq.)}$$

$$\frac{\dot{\widetilde{T}}}{\widetilde{T}} = \frac{R}{C_p}\frac{\dot{\pi}}{\pi} + \frac{Q}{C_p T}, \qquad \textit{(Thermodynamic eq.)}$$

QE perfect gaz law $\pi = \widetilde{\rho} R \widetilde{T} \Rightarrow$

$$\frac{\dot{\pi}}{\pi} + \frac{C_p}{C_v} D_3 = \frac{Q}{C_v T}, \qquad \textit{(QE momentum constraint eq.)}$$

Mass-based vertical coordinate transformation

Choosing π as a vertical coordinate *[Laprise 92]*

Coordinate transformation $z \to \pi \to \eta \in [0,1]$: $\pi = A(\eta) + B(\eta)\pi_s$ and $m = \partial\pi/\partial\eta$,

$$\frac{\partial \phi}{\partial \eta} = -\frac{1}{\tilde{\rho}}\frac{\partial \pi}{\partial \eta} = -m\frac{R\tilde{T}}{\pi}$$

[Kasahara] mass continuity equation $\Rightarrow$

$$\frac{\partial m}{\partial t} + \nabla\cdot(mV) + \frac{\partial}{\partial \eta}(m\dot{\eta}) = 0$$

with $\dot{\eta}(0) = \dot{\eta}(1) = 0 \Rightarrow$

$$\begin{aligned} \dot{\pi} &= V\cdot\nabla\pi - \int_0^{\eta} \nabla\cdot(mV)\,\mathrm{d}\eta' \\ m\,\dot{\eta} &= B(\eta)\int_0^1 \nabla\cdot(mV)\,\mathrm{d}\eta' - \int_0^{\eta}\nabla\cdot(mV)\,\mathrm{d}\eta' \end{aligned}$$

Choosing π as a vertical coordinate *[Laprise 92]*

Coordinate transformation rules :

$$\begin{aligned} \frac{\partial}{\partial z} &= -g\frac{\widetilde{\rho}}{m}\frac{\partial}{\partial \eta} \\ \nabla_z &= \nabla + \frac{\widetilde{\rho}}{m}\nabla\phi\frac{\partial}{\partial \eta} \\ \frac{\mathrm{d}}{\mathrm{d}t} &= \frac{\partial}{\partial t} + V\cdot\nabla + \dot{\eta}\frac{\partial}{\partial \eta} \end{aligned}$$

Momentum equations $\Rightarrow$

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} &= -\frac{\widetilde{\rho}}{\rho}\left[\frac{1}{m}\frac{\partial p}{\partial \eta}\nabla\phi + \frac{1}{\rho}\left(1+\frac{\rho-\widetilde{\rho}}{\widetilde{\rho}}\right)\nabla p\right] + \mathcal{F}_V \\ \frac{\mathrm{d}w}{\mathrm{d}t} &= g\frac{\widetilde{\rho}}{\rho}\left[\frac{1}{m}\frac{\partial(p-\pi)}{\partial \eta} - \frac{\rho-\widetilde{\rho}}{\widetilde{\rho}}\right] + \mathcal{F}_w \end{aligned}$$

QE mass-based dynamical continuous system

Prognostic equations :

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -e^{\kappa q}\left[\left(1+\frac{1}{m}\frac{\partial q}{\partial \eta}\right)\nabla\phi + R\widetilde{T}\left(\frac{\nabla\pi}{\pi}+\nabla q\right)\right] + \mathcal{F}_V$$

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{g}{\kappa}\left[\frac{\pi}{m}\frac{\partial\left(e^{\kappa q}-1\right)}{\partial\eta} + \kappa\left(e^{\kappa q}-1\right)\right] + \mathcal{F}_w$$

$$\frac{\mathrm{d}\widetilde{T}}{\mathrm{d}t} = \frac{R\widetilde{T}}{C_p}\frac{\dot{\pi}}{\pi} + \frac{\mathcal{Q}e^{-\kappa q}}{C_p}$$

$$\frac{\partial\pi_s}{\partial t} = -\int_0^1 \nabla\cdot(mV)\,\mathrm{d}\eta'$$

QE mass-based dynamical continuous system

Diagnostic equations :

$$\widetilde{\rho} = \frac{\pi}{R\widetilde{T}}$$

$$\nabla\phi = \nabla\phi_s + \int_\eta^1 \nabla\left(m\frac{R\widetilde{T}}{\pi}\right) \mathrm{d}\eta'$$

$$\frac{\dot{\pi}}{\pi} = V\cdot\frac{\nabla\pi}{\pi} - \frac{1}{\pi}\int_0^\eta \nabla\cdot(mV)\,\mathrm{d}\eta'$$

$$m\,\dot{\eta} = B(\eta)\int_0^1 \nabla\cdot(mV)\,\mathrm{d}\eta' - \int_0^\eta \nabla\cdot(mV)\,\mathrm{d}\eta'$$

$$D_3 = \nabla\cdot V + \frac{\widetilde{\rho}}{m}\nabla\phi\cdot\frac{\partial V}{\partial\eta} - g\frac{\widetilde{\rho}}{m}\frac{\partial w}{\partial\eta}$$

QE mass-based dynamical continuous system

QE momentum constraint :

$(V, w, \widetilde{T}, \pi_s) \rightarrow$ prognostic variables. $\dot{q} = 0 \Rightarrow q$ is diagnostically determined in such a way the velocity fields satisfy

$$D_3 + \frac{C_v}{C_p}\frac{\dot{\pi}}{\pi} = \frac{Q}{C_p}\, e^{-\kappa q} \qquad \Rightarrow$$

$$D - \frac{C_v}{C_p}\frac{1}{\pi}\int_0^{\eta}(m\,D)\,\mathrm{d}\eta' + \overbrace{\left(-g\,\frac{\widetilde{\rho}}{m}\frac{\partial w}{\partial \eta}\right)}^{d} + \underbrace{\frac{\widetilde{\rho}}{m}\nabla\phi\cdot\frac{\partial V}{\partial \eta} + \frac{C_v}{C_p}\left[V.\frac{\nabla\pi}{\pi} - \frac{1}{\pi}\int_0^{\eta} V.\nabla m\,\mathrm{d}\eta'\right]}_{Y} = \frac{Q}{C_p}\, e^{-\kappa q}$$

$\Rightarrow$

$$\left(1 - \frac{C_v}{C_p}\mathcal{S}\right) D + d + Y = \frac{Q}{C_p}\, e^{-\kappa q}$$

Use of $dl = d + \Upsilon$ as prognostic variable in lieu of w

Vertical momentum eq. $\Rightarrow$

$$\frac{\mathrm{d}dl}{\mathrm{d}t} = -\frac{g^2}{R\widetilde{T}}\frac{\pi}{m}\frac{\partial}{\partial\eta}\left[\left(\frac{\pi}{m}\frac{\partial}{\partial\eta}+\kappa\right)\left(\frac{e^{\kappa q}-1}{\kappa}\right)+\frac{\mathcal{F}_w}{g}\right] + \left(\frac{\pi}{m}\frac{\partial V}{\partial\eta}\right)\cdot\frac{g\nabla w}{R\widetilde{T}} + (dl-\Upsilon)(D-D_3)+\dot{\Upsilon},$$

QE momentum constraint $\Rightarrow$

$$\left(1-\frac{C_v}{C_p}\mathcal{S}\right)D + dl = \frac{\mathcal{Q}}{C_p}\,e^{-\kappa q}$$

Diagnostic for w $\Rightarrow$

$$w = w_s + \int_\eta^1 m\frac{R\widetilde{T}}{g\pi}(dl-\Upsilon)\,\mathrm{d}\eta'$$

Semi-implicit time discretization method

Basic concept

Defining $X = [V, w, \widetilde{T}, \widetilde{q}_{\mathsf{S}}]$ the vector of prognostic variables and $Z = [X, q]$ the state vector, the QE system can be symbolically written as

$$\frac{\partial X}{\partial t} = \mathscr{A}(X) + \hat{\mathscr{M}}_{x}(Z) \equiv \mathscr{M}(Z)$$

together with the QE momentum constraint :

$$\hat{\mathscr{M}}_{\mathsf{q}}(X) = 0$$

The physical contributions are not taken into account.

A proposed SI time discretization

Basic concept

As a classical 3-TL SI Eulerienne approach :

$$\frac{X^{+}-X^{-}}{2\Delta t}=\mathscr{M}\left(Z^{0}\right)-\mathscr{L}^{*}.Z^{0}+\frac{\mathscr{L}^{*}.Z^{-}+\mathscr{L}^{*}.Z^{+}}{2},$$

with

$$\hat{\mathscr{M}}_{\mathbf{q}}\left(X^{+}\right)=0$$

The prognostic fields discretely satisfy the non-linear QE momentum constraint at $t+\Delta t$.

A proposed SI time discretization

Basic concept

The SI scheme is combined with a Newton-Raphson-like iterative treatment of the QE momentum constraint, it yields

$$\frac{X^{+(\nu)} - X^{-}}{2\Delta t} = \mathscr{M}\left(Z^{0}\right) - \mathscr{L}^{*}.\,Z^{0} + \frac{\mathscr{L}^{*}.\,Z^{-} + \mathscr{L}^{*}.\,Z^{+(\nu)}}{2},$$

with

$$\hat{\mathscr{M}}_{\mathbf{q}}\left[X^{+(\nu-1)}\right] - \mathscr{C}^{*}.\,X^{+(\nu-1)} + \mathscr{C}^{*}.\,X^{+(\nu)} = 0$$

1. Stability is controlled by the magnitude of the explicit residual terms ($\mathscr{M} - \mathscr{L}^{*}$).
2. Convergence is determined by the magnitude of the non-linear residual term ($\hat{\mathscr{M}}_{\mathbf{q}} - \mathscr{C}^{*}$) of QE momentum constraint.

Semi-implicit time discretization method

Iterative linear implicit problem

For $\nu \in [1, N_{\text{stop}}]$,

$$X^{+(\nu)} - \Delta t\, \mathscr{L}^*. Z^{+(\nu)} = \mathscr{X}^\bullet$$

$$\mathscr{C}^*. X^{+(\nu)} = \mathscr{R}^{+(\nu-1)}$$

1. Our choice of $\mathscr{L}^*$ and $\mathcal{C}^*$ consists in linearizing the full dynamical system around a resting stationary, isothermal and horizontally homogeneous reference-state X^*
2. N_{stop} is determined in such a way that the QE constraint $\hat{\mathscr{M}}_{\mathsf{q}}(Z^{+(N_{\text{stop}})}) \simeq 0$ is fulfilled at an acceptable degree of accuracy (machine precision is typically required).

SI time-discretized system

Iterative linear implicit problem

$$D^{+(\nu)} + \Delta t\, RT^* \nabla^2 \left[\mathbf{G}^* \widetilde{T}_*^{+(\nu)} + \widetilde{q}_S^{+(\nu)} + q^{+(\nu)} \right] = \nabla \cdot \mathscr{V}^\bullet$$

$$dl^{+(\nu)} + \Delta t \frac{g}{rH_*} \mathbf{L}_\kappa^* \, q^{+(\nu)} = \mathscr{D}^\bullet,$$

$$\widetilde{T}_*^{+(\nu)} + \Delta t\, \kappa \mathbf{S}^* D^{+(\nu)} = \mathscr{T}^\bullet,$$

$$\widetilde{q}_S^{+(\nu)} + \Delta t\, \mathbf{N}^* D^{+(\nu)} = \mathscr{P}_s^\bullet$$

with

$$\mathbf{S}_\kappa^* D^{+(\nu)} + dl^{+(\nu)} = \mathscr{R}^{+(\nu-1)} \equiv (\mathbf{S}_\kappa^* - \mathbf{S}_\kappa)\, D^{+(\nu-1)},$$

SI time-discretized system

Algebraical constraints on the vertically discrete linear model

Considering $\mathbf{G}^*$, $\mathbf{S}^*$, $\mathbf{N}^*$, and $\mathbf{L}_v^*$ as already existing operators for EE system,

1. $\mathbf{S}_\kappa^* = [\mathbf{I} - (1-\kappa)\mathbf{S}]$ is an invertible integral matrix.
2. $\mathbf{L}_\kappa^* = \mathbf{S}_\kappa^* \mathbf{L}_v^*$ is akin as a tridiagonal vertical Laplacian operator
3. The following identity should be satisfied :

$$\mathbf{L}_v^*[\mathbf{G}^*\mathbf{S}^* + (1/\kappa)\mathbf{N}^*] = -\mathbf{T}^*$$

   where $\mathbf{T}^*$ is a tridiagonal vertical averaging operator similar to the one describes in BHBG95. hereafter $\mathbf{G}^*\mathbf{S}^* + (1/\kappa)\mathbf{N}^* \equiv \mathbf{K}^*$.

Resolution of the iterative linear implicit problem

Elimination procedure

$$\left[\mathbf{I} - \Delta t^2 N_*^2 \mathbf{K}^* \nabla_*^2\right] D^{+(\nu)} + \frac{1}{\tau^*} \nabla_*^2 q^{+(\nu)} = D^\bullet,$$

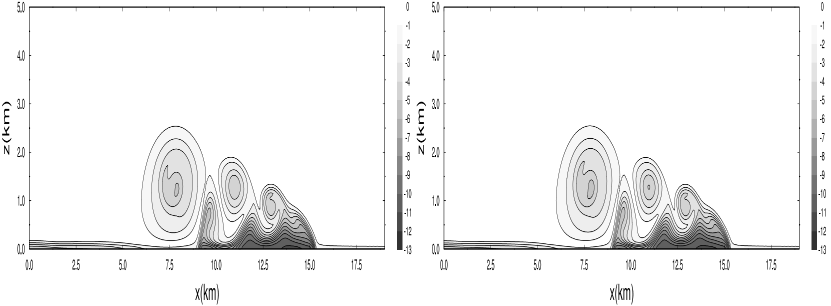
$$D^{+(\nu)} - \frac{1}{\tau^* r} \mathbf{L}_v^* q^{+(\nu)} = \mathbf{S}_\kappa^{*\,\text{-}1} \left[\mathscr{R}^{+(\nu-1)} - \mathscr{D}^\bullet\right],$$

$\Rightarrow$ For $\nu \in [1, N_{\text{stop}}]$

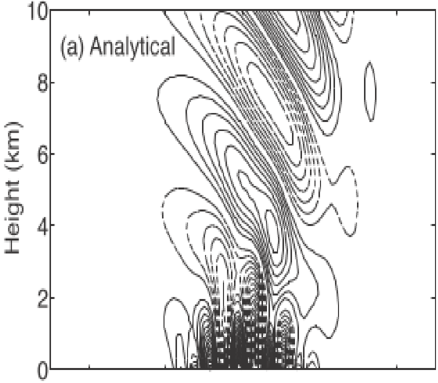
$$\left[\mathbf{L}_v^* + \left(r\mathbf{I} + \Delta t^2 N_*^2\, \mathbf{T}^*\right) \nabla_*^2\right] D^{+(\nu)} = D^{\bullet\bullet} + r\nabla_*^2 \mathbf{S}_\kappa^{*\,\text{-}1} \left(\mathbf{S}_\kappa^* - \mathbf{S}_\kappa\right) D^{+(\nu-1)},$$

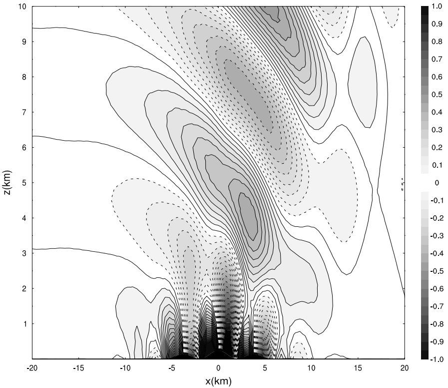
Idealized 2D test-case

Density current test case: Distributions of perturbations of potential temperature for the cold bubble experiment : spacing $\Delta x = 50\,\text{m}$ and $\Delta z \approx 50\,\text{m}$, with $\Delta t = 0.25\,\text{s}$. EE solution (left) and QE solution (right) at $t = 900\,\text{s}$

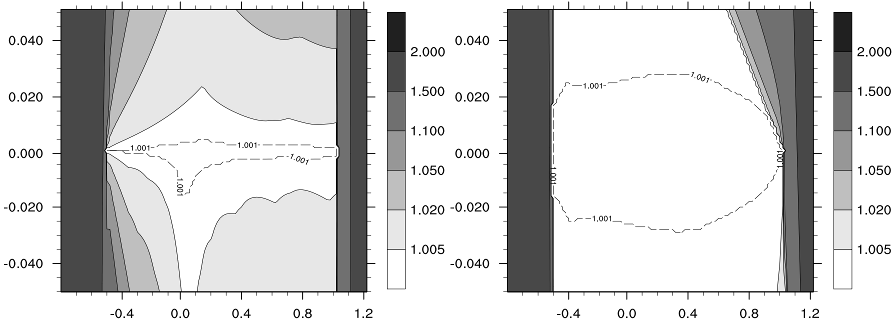


Schär mountain test case : Numerical solution of vertical velocity ($\mathrm{m\,s^{-1}}$) at $t = 12\,\mathrm{h}$, with $\Delta t = 4\,\mathrm{s}$, $\Delta x = 500\,\mathrm{m}$, and $\Delta z \simeq 300\,\mathrm{m}$.





stability as function a uniform slope s (%) and the thermal non-linear residual factor $\alpha = (T - T^*)/T^*$ actual 3-TL SI EE (left) 3-TL SI QE (right)



Summary and perspectives

1. QE soundproof system can be readily blended into the existing HPE/EE ARPEGE/AROME Dynamical kernel using Laprise approach.
2. A semi-implicit time discretization of such a soundproof system is now possible in a similar framework as in AROME (extension to ICI scheme is also possible).
3. Numerical testing with SI QE system seems to present a better stability property than SI EE system while maintaining a similar degree of accuracy.