Royal Netherlands Meteorological Institute
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Optimal linearization trajectories for linear models

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Nowadays linear models play a crucial role in important components of NWP. Examples of such application are Ensemble Forecasting and Variational Data-assimilation. Nonlinear model behavior however can have severe implications for the usefulness of these linear models. We show that many types of nonlinearities can successfully be taken into account in linear models by a suitable choice of the linearization trajectory.

1. Introduction

Suppose we are dealing with differential equations of the form

$$d/dt\ \mathbf{X} = s(\mathbf{X},\mathbf{X})+b(\mathbf{X})+c \qquad (1)$$

with s and b symmetric and linear operators respectively and forcing term c. The evolution of perturbation $\mathbf{x}$ of $\mathbf{X}$ is given by

$$d/dt\ \mathbf{x} = 2s(\mathbf{X},\mathbf{x})+b(\mathbf{x})+s(\mathbf{x},\mathbf{x}) \quad (2)$$

Here $\mathbf{J}$ with $\mathbf{J}[\mathbf{X}]\mathbf{x} = 2s(\mathbf{X},\mathbf{x})+b(\mathbf{x})$ is the Jacobian of (1) evaluated along the trajectory $\mathbf{X}(t)$. Note that for the tangent linear approximation the term $s(\mathbf{x},\mathbf{x})$ is neglected in (2).

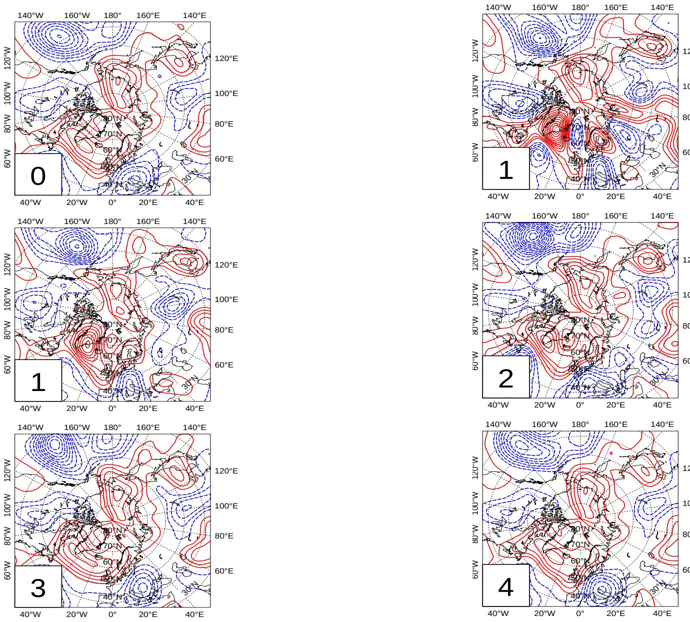

Figure A. *Streamfunction perturbation at 500 hPa nonlinearly evolved with a 3-level quasi-geostrophic model (top left), the standard tangent linear evolution (top right), and four iterations (1-4) of the iterative relinearization method (4). Positive (negative) values are plotted solid (dashed).*

2. Relinearization

The key observation here is that the exact time evolution of perturbations $\mathbf{x}$ of $\mathbf{X}$ as given by (2) can also be written as

$$d/dt\ \mathbf{x} = \mathbf{J}[\mathbf{X}+\mathbf{x}/2]\mathbf{x} \qquad (3)$$

The trajectory $\mathbf{X}+\mathbf{x}/2$ is referred to as the optimal linearization trajectory.

3. Computation

Let $\mathbf{M}[\mathbf{Y}]$ be the propagator of the regular tangent linear system linearized around trajectory $\mathbf{Y}$. The optimal linearization trajectory is determined by applying the following iteration procedure

$$\mathbf{x}^k(t) = \mathbf{M}[\mathbf{X}+\mathbf{x}^{k-1}/2]\ \mathbf{x}(0) \qquad (4)$$

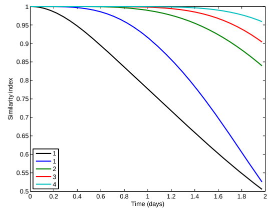

Figure B. *Average similarity index (20 cases) between $\mathbf{x}^k(t)$ and the true perturbation $\mathbf{x}(t)$ for 3-level quasi-geostrophic model runs. The black line is the standard tangent linear model ($\mathbf{x}^0(t) = 0$)*

4 Results

4.1 Prediction

Figure A shows for a single case and figure B for an average of 20 cases how the iteration procedure quickly converges, yielding the optimal linearization trajectory. By employing this trajectory, the tangent linear model produces identical results to the nonlinear evolution of $\mathbf{x}(0)$:

$$\mathbf{x}(t) = \mathbf{M}[\mathbf{X}+\mathbf{x}/2]\ \mathbf{x}(0) \qquad (5)$$

4.2 Estimation

Here one is interested to retrieve the analysis increment that produces a given forecast error. Figure C shows how the optimal linearization trajectory helps in finding the analysis increment. The shortcoming of the standard tangent linear model in this is obvious (black lines). See also the quality of the nonlinear update starting from $\mathbf{X}(0)+\mathbf{x}^1(0)$.

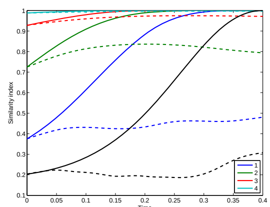

Figure C. *Averaged similarity index (50 cases) between the true perturbation trajectory $\mathbf{x}(t)$ and $\mathbf{x}^k(t)$ (nonlinearly evolved $\mathbf{x}^k(t)$) is given by solid (dashed) lines. The Lorenz 96 model is used with a lead time of 0.4 units (2 days).*

4. Conclusion

- By modifying the linearization trajectory of the tangent linear model it is possible to produce identical forecasts as with the nonlinear model

- Preliminary results indicate that use of the optimal linearization trajectory, instead of nonlinear updates, improves the performance of standard 4D-VAR.

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