New "Rotated/Tilted Mercator" geometry in Aladin

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ABSTRACT

A new rotated/tilted Mercator geometry is introduced. The associated computations are developped. These computations involve formulae for :
- the direct geometric transformation
- the reverse geometric transformation
- the map factor
- the transformation of vectors (so-called "compass").

Finally, coding and user aspects are shortly discussed.

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1 Introduction

The rotated/tilted Mercator geometry has the advantage of allowing a focus on any part of the sphere (polar area, extra-tropical area, tropical-area) in a single formalism, altogether with a simple form of the map factor. Especially, since the map factor is only a fonction of the second coordinate y, a strategy in which the map factor is taken into account in the semi-implicit scheme becomes easy to apply (so-called "`LSIDG`" strategy in ARPEGE's jargon). This is not the case for normal (non rotated) Lambert projections. This strategy may become necessary for large domain extensions as considered in HIRLAM group for NWP applications.
The basic characteristics of the geometry described in this paper are as follows :

- A reference point of the projection is chosen and defined by its geographical coordinates $P_0 = (\lambda_0, \theta_0)$. The reference point of the Mercator projection is the one that is on the circle of tangence of the projection cylinder and of the sphere, and which is at the opposite longitude than the one of the cutting line of the cylinder.
- A rotation is applied in order to bring the reference point on the rotated equator, rotating this point along the meridian defined by $\lambda = \lambda_0$. Hence, after the rotation, the reference point lies in the equator and the local north direction is left unchanged.
- A tilting of the sphere around the new origin (i.e. the reference point) with an angle β is optionally applied in order to allow elongated countries to be represented as lying along the transformed Equator.
- A standard Mercator projection is finally applied for the rotated/tilted sphere obtained after the previous steps, and with the reference point as origin. As a consequence, the local north for the reference point makes an angle β with the Oy axis in the projected plane.
- The LAM domain is taken as symmetric around the reference point in both x and y directions. As a consequence, the reference point lies at the center of the plane rectangular LAM domain. This latter is the point where the best focus is desired for the LAM application. Since the reference point of the projection and the center of the domain are identical, no clear distinction is made between these two concepts in this paper, except in section 9 when the comparison with current tilted Lambert projections is discussed.

Finally, we see that the reference point necessarily lies at the center of the rectangular Mercator domain, and that at this reference point, the northward direction is tilted with an angle β with the Oy axis (in the most common case $\beta = 0$, the northward direction is parallel to the $0y$ axis for the reference point).

Remark : In this document, a clear distinction is made between the "rotated" geometry, which denotes the geometry obtained after the first rotation, and the "rotated/tilted" geometry, which denoted the geometry obtained after the first rotation and the tilting rotation. Thoughout all the following, the two terms "rotated" and "rotated/tilted" thus have a disctinct meaning, not to be confused.

The geographical Northen Pole is denoted NP and the geographical Southern Pole is denoted SP throughout.

2 Definition of the geometry

The choice of a particular projected LAM domain can be decomposed into two main steps : the definition of the geometry (i.e. projection), and the definition of the domain boundaries *in this geometry*.

2.1 Choice of the projection

As a first step, a reference point must be chosen. This is the point where the focus is desired for the LAM application. This reference point is defined by its geographical coordinates (λ_0, θ_0). As a second step, a tilting angle $\beta \in [-\pi, \pi]$ must be specified. The value of β is the angle between the local geographical northward direction and the Oy axis in the projected plane (see section 2.7 for the sign convention)

2.2 Particular case $\theta_0 = \pm\pi/2$

If the reference point is chosen exactly on one of the two geographical poles (i.e. $\cos\theta_0 = 0$), the longitude of the reference point is undetermined. An arbitrary longitude λ_0 can then be chosen. The effect of chosing $\lambda_0 \neq 0$ is simply to rotate the domain around the reference point, then allowing the reference meridian $\lambda = \lambda_0$ to be parallel to the Oy axis of the Mercator map, as it would be if θ_0 was very close, but not equal to $\pm\pi/2$. It should be noted that in the case $\cos\theta_0 = 0$, a similar effect can also be obtained with the "tilting" rotation, since chosing a tilting angle $\beta \neq 0$ provides the same result. As a consequence, when $\cos\theta_0 = 0$, one of the two degrees of freedom (λ_0, β) is redundant. Such a redundance is not a problem and can be kept in the code. Therefore, when $\cos\theta_0 = 0$ is chosen, a reference longitude λ_0 <u>must</u> be provided as well as a tilting angle β, and both parameter are active, in a way to insure the continuity with the $\cos\theta_0 \neq 0$ case.

2.3 Choice of the domain boundaries

The choice of the domain boundaries is made by specifying the number of points in each of the Ox and Oy directions, and the uniform resolutions Δx and Δy in the projected plane. The specification of the domain boundaries is not really related to the definition of the geometry, and is not discussed in more details in this paper.

2.4 First rotation

In this section we essentially use the formulae presented in the 'Documentation ARPEGE', Chapter 7 : "La Sphère Transformée". These formulae allow to compute the spherical coordinates of a point in a rotated frame, for which the pole of spherical coordinates is at the point of geographical spherical coordinates (λ_p, θ_p), and for which the rotated longitude of the geographical North Pole is zero. We first redefine the formulae as a function of the reference point (λ_0, θ_0) instead of (λ_p, θ_p), and we show that the resulting formulae are valid for postive and negative values of θ_0. Then the reverse formulae are derived.

Case $\theta_0 > 0$:

Let (λ_0, θ_0) be the geographical coordinates of the reference point, noted P_0. We first assume that θ_0 is strictly positive, that is, the reference point P_0 is in the northern hemisphere. The rotation that we want to use for our geometry must bring P_0 at the equator, and at the origin of longitudes. The point which will become the "upper" pole after the rotation is noted U' (the "upper" pole U' is the the pole of the rotated sphere that is encountered first when travelling northward starting from P_0 on the meridian circle passing through P_0). The geographical coordinates of U' are then given by (see graphics) :

$$\lambda_p = \lambda_0 + \pi \tag{1}$$

$$\theta_p = \frac{\pi}{2} - \theta_0 \tag{2}$$

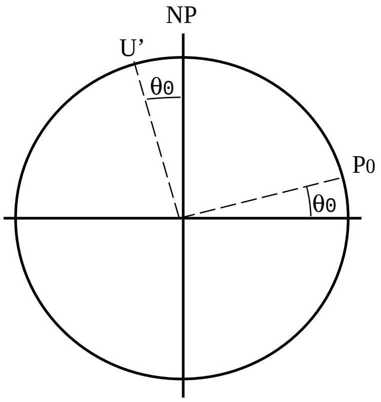

FIG. 1 – *The various points important for the first rotation when $\theta_0 > 0$, as seen (before the first rotation) from $90°$ of longitude west of P_0.*

We thus have the following relationships :

$$\sin\theta_p = \cos\theta_0 \tag{3}$$

$$\cos\theta_p = \sin\theta_0 \tag{4}$$

$$\sin(\lambda - \lambda_p) = -\sin(\lambda - \lambda_0) \tag{5}$$

$$\cos(\lambda - \lambda_p) = -\cos(\lambda - \lambda_0) \tag{6}$$

Let (λ, θ) be the geographical coordinates of an arbitrary point on the sphere. The coordinates (λ', θ') of this point in the frame which has for "upper" pole the point (λ_p, θ_p) and for which the geographical Northen Pole is on the origin transformed meridian ($\lambda'_{\mathrm{NP}} = 0$) is given by (cf. Documentation ARPEGE, Chapter 7 : "La Sphère Transformée") :

$$\theta' = \arcsin\left[\sin\theta_p \sin\theta + \cos\theta_p \cos\theta \cos(\lambda - \lambda_p)\right] \tag{7}$$

$$\cos\lambda' = \left(\frac{1}{\cos\theta'}\right)\left[\cos\theta_p \sin\theta - \sin\theta_p \cos\theta \cos(\lambda - \lambda_p)\right] \tag{8}$$

$$\sin\lambda' = -\left(\frac{1}{\cos\theta'}\right)\left[\cos\theta \sin(\lambda - \lambda_p)\right] \tag{9}$$

which leads, after substitution of the previous relationships, to :

$$
\begin{aligned}
\theta' &= \arcsin\left[\cos\theta_0 \sin\theta - \sin\theta_0 \cos\theta \cos(\lambda-\lambda_0)\right] &(10)\\
\cos\lambda' &= \left(\frac{1}{\cos\theta'}\right)\left[\sin\theta_0 \sin\theta + \cos\theta_0 \cos\theta \cos(\lambda-\lambda_0)\right] &(11)\\
\sin\lambda' &= \left(\frac{1}{\cos\theta'}\right)\left[\cos\theta \sin(\lambda-\lambda_0)\right] &(12)
\end{aligned}
$$

These formulae are exactly matching our purpose since in the case where P_0 is in the norther hemisphere, we precisely wish that the geographical Northern Pole lies at $\lambda' = 0$, that is, for a LAM, on the origin meridional axis, in order that it can be represented properly in the LAM Cartesian domain afterward. This remark is mainly valid when θ_0 is close from $\pi/2$ (i.e. the reference poit P_0 is close from the Northen Pole, but by convenience and continuity, it may be considered as to apply for any positive values of θ_0.

Case $\theta_0 < 0$:

In the case where $\theta_0 < 0$, some of the steps in the previous reasoning are modified, and thus the derivation must be reiterated. However, it will be found that the relationships for this case are finally the same as for the previous case, thus allowing a single set of relationships for all cases. If $\theta_0 < 0$ (P_0 is in the southern hemisphere), the rotation that we want to describe for our geometry still must bring P_0 at the equator, and vanishing longitude. Applying this rotation, the geographical coordinates of the new "upper" pole will thus be :

$$
\begin{aligned}
\lambda_p &= \lambda_0 &(13)\\
\theta_p &= \frac{\pi}{2} + \theta_0 &(14)
\end{aligned}
$$

We thus have the following relationships :

$$
\begin{aligned}
\sin\theta_p &= \cos\theta_0 &(15)\\
\cos\theta_p &= -\sin\theta_0 &(16)\\
\sin(\lambda-\lambda_p) &= \sin(\lambda-\lambda_0) &(17)\\
\cos(\lambda-\lambda_p) &= \cos(\lambda-\lambda_0) &(18)
\end{aligned}
$$

For an arbitrary point on the sphere with geographical coordinates (λ, θ), the coordinates (λ', θ') in the frame which has for "upper" pole the point (λ_p, θ_p) and for which the geographical Northen Pole is on the origin transformed meridian ($\lambda'_{\mathrm{NP}} = 0$) is still given by (7)-(9) :
which leads, after substitution of the previous relationships, to :

$$
\begin{aligned}
\theta'_{\mathrm{prov}} &= \arcsin\left[\cos\theta_0 \sin\theta - \sin\theta_0 \cos\theta \cos(\lambda-\lambda_0)\right] &(19)\\
\cos\lambda'_{\mathrm{prov}} &= \left(\frac{1}{\cos\theta'_{\mathrm{prov}}}\right)\left[-\sin\theta_0 \sin\theta - \cos\theta_0 \cos\theta \cos(\lambda-\lambda_0)\right] &(20)\\
\sin\lambda'_{\mathrm{prov}} &= \left(\frac{1}{\cos\theta'_{\mathrm{prov}}}\right)\left[-\cos\theta \sin(\lambda-\lambda_0)\right] &(21)
\end{aligned}
$$

where the subscript "prov" stands for "provisional" as detailed now : we see that compared to the case $\theta_0 > 0$, these formulae formally give identical latitudes θ', but opposite longitudes λ'. However, the rotation described by these formulae is not exactly the one that we ideally want when P_0 is in the southern hemisphere : in this case, we would prefer the geographical Southern Pole to be located on the $\lambda' = 0$ axis, for reasons symmetric to the ones explained above for $\theta_0 > 0$. As a direct consequence, the geographical Northern Pole should be located at the two edges $\lambda' = \pm\pi$ of the longitude domain, while it is located at $\lambda' = 0$ with the formulae (19)-(21). Therefore for the case $\theta_0 < 0$ we have to apply a shift in longitudes, in order to bring the geographical Southern Pole at the origin of meridional location $\lambda' = 0$. By construction, this shift in longitudes is defined by :

$$\theta' = \theta'_{\text{prov}} \tag{22}$$

$$\lambda' = -\lambda'_{\text{prov}} \tag{23}$$

Finally substituting these two latter relationships into (19)-(21) yields :

$$\begin{aligned}
\theta' &= \arcsin\left[\cos\theta_0 \sin\theta - \sin\theta_0 \cos\theta \cos(\lambda - \lambda_0)\right] \\
\cos\lambda' &= \left(\frac{1}{\cos\theta'}\right)\left[\sin\theta_0 \sin\theta + \cos\theta_0 \cos\theta \cos(\lambda - \lambda_0)\right] \\
\sin\lambda' &= \left(\frac{1}{\cos\theta'}\right)\left[\cos\theta \sin(\lambda - \lambda_0)\right]
\end{aligned}$$

which is formally identical to (10)–(12). Therefore, the same set of formulae can be used for $\theta_0 > 0$ as well as for $\theta_0 < 0$.

Case $\theta_0 = 0$:

If $\theta_0 = 0$ (that is if the reference point P_0 is on the geographical Equator), then the direct formulae of the rotation (10)-(12) readily apply and give :

$$\theta' = \theta \tag{24}$$

$$\lambda' = (\lambda - \lambda_0) \tag{25}$$

which is the expected result.

Results with the first rotation

In any of the above cases for θ_0, the reference point P_0 has its new coordinates given by ($\lambda' = 0, \theta' = 0$).
Here we present in Fig. 2 and Fig. 3, two maps in the modified lat-lon coordinates. The first one is obtained by specifying $(\lambda_0, \theta_0) = (-68°, +8°)$ (that is, P_0 lies in the northern hemisphere), while the second is for $(\lambda_0, \theta_0) = (-68°, -8°)$ (that is, P_0 lies in the southern hemisphere). The results of the discussion above are clearly seen in these to figures : in any case, the geographical pole which lies at the meridional origin $\lambda' = 0$ is the one which is in the same hemisphere as P_0, as desired.

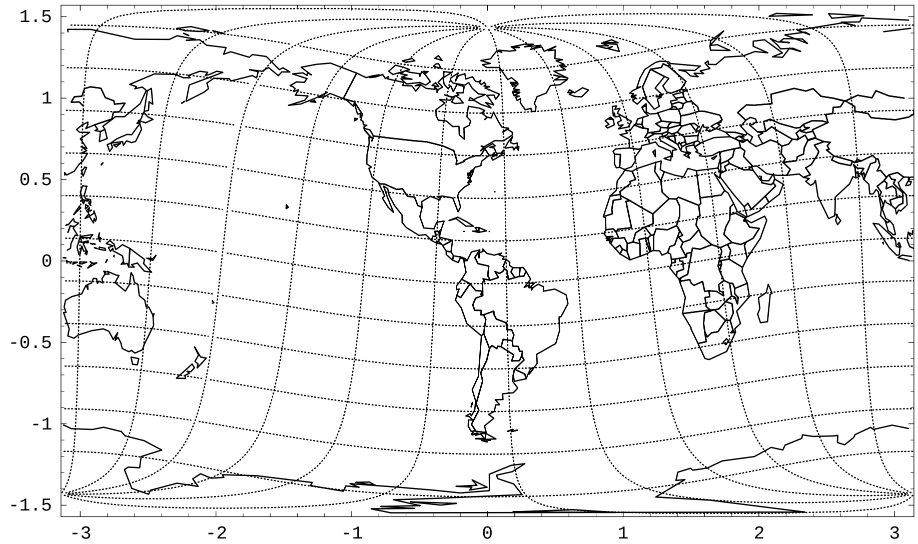

FIG. 2 – *Map in lat-lon obtained with the rotation described in the text, for* $(\lambda_0, \theta_0) = (-68°, +8°)$.

2.5 Safeguards and remarks for the first rotation

Safeguard : *For* $\theta' = 0$ *(that is for the two points which are brought to the two new poles), formulae (11)–(12) have to be ignored, and the rotated longitude* λ' *has to be defined conventionally (e.g.* $\lambda' = 0$*). This conventional definition has of course no impact on the results of subsequent geometric transformations.*

N.B.1 : In (10) the 'arcsin' function is sufficient to uniquely determine θ' provided that the image of [-1,1] by 'arcsin' is [- $\pi/2$, $\pi/2$], which is usually the case for standard built-in functions of arithmetic packages.

N.B.2 : The rotated longitude λ' is readily determined from (11) and (12) through :

$$\lambda' = \mathrm{Arg}(\cos\lambda' + i\sin\lambda') \tag{26}$$

In case the built-in function 'Arg' is not available, the latter definition must be replaced by the more standard form :

$$\lambda' = \arccos(\cos\lambda')\,.\,\mathrm{Sign}(\sin\lambda') \tag{27}$$

where Sign(0) must absolutely be equal to 1 (some arithmetic packages provide zero).

2.6 Justifications for a "tilting" rotation

A tilting of the geometry is now introduced. This tilting consists in a rotation around the origin of the previously obtained rotated sphere ($\lambda' = 0, \theta' = 0$). It should be outlined that usually, such a tilting is not required, especially if domains with an aspect ratio close to 1 are to be used. However, for some very elongated countries or interest areas, it may be wishable to also have an elongated LAM domain, and if the elongation direction is close to the y direction of the projection,

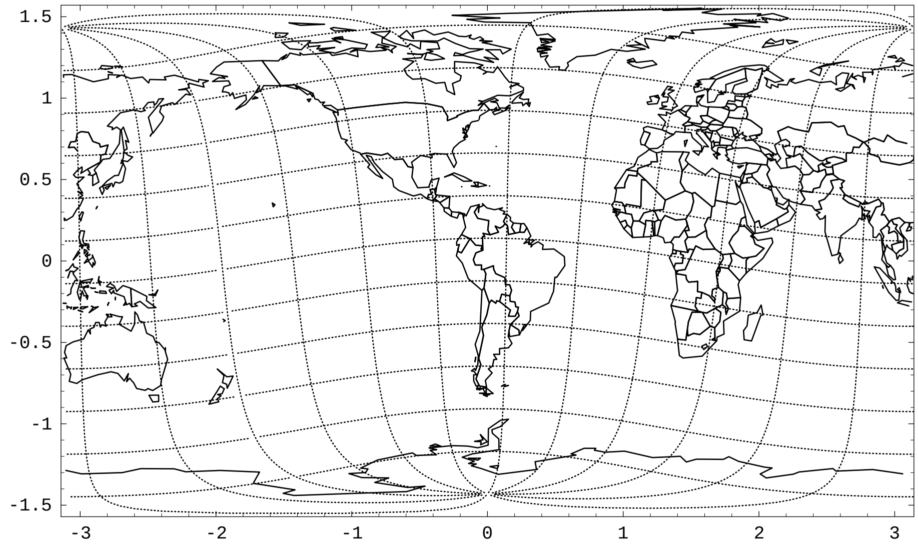

FIG. 3 – *Map in lat-lon obtained with the rotation described in the text, for* $(\lambda_0, \theta_0) = (-68°, -8°)$.

then the non-tilted strategy is not optimal because it implies unnecessarily large variations of the map factor. In such cases a tilting of the sphere before the Mercator projection allows to bring the elongated direction of the domain along the Equator of the rotated/tilted sphere and therefore, allows to minimize the variations of the map factor for the projected domain. A good example of this situation is the case of Chile, as depicted in Fig. 4 below.
The tilting could also be required and justified by more physical reasons, e.g. in order to avoid the appearance of undesired high mountains near the corners of the domain, or to make the left edge of the domain (in the northen hemisphere) more perpendicular to the main advecting eastward flux .

2.7 "Definition of the tilting" rotation

The tilting rotation is defined by an angle $\beta \in [-\pi, \pi]$. The angle of the tilting rotation is counted positively counter-clockwise when looking to the rotated origin $(\lambda' = 0, \theta' = 0)$ from the sky. The tilting rotation can be mathematically expressed in a similar way as for the first rotation, by simply examining the point UP'' which will be brought at the upper pole of the rotated/tilted sphere. The coordinates of UP'' in the rotated geometry are given by :

$$\lambda'_p = +\pi/2\,, \quad \theta'_p = (\pi/2 - \beta) \quad \text{if} \quad \beta > 0 \tag{28}$$
$$\lambda'_p = -\pi/2\,, \quad \theta'_p = (\pi/2 + \beta) \quad \text{if} \quad \beta < 0 \tag{29}$$

Case $\beta > 0$:

Applying (7)-(9) to (28), the coordinates (in the rotated/tilted geometry) of a point which has coordinates (λ', θ') in the rotated geometry will be, after the tilting :

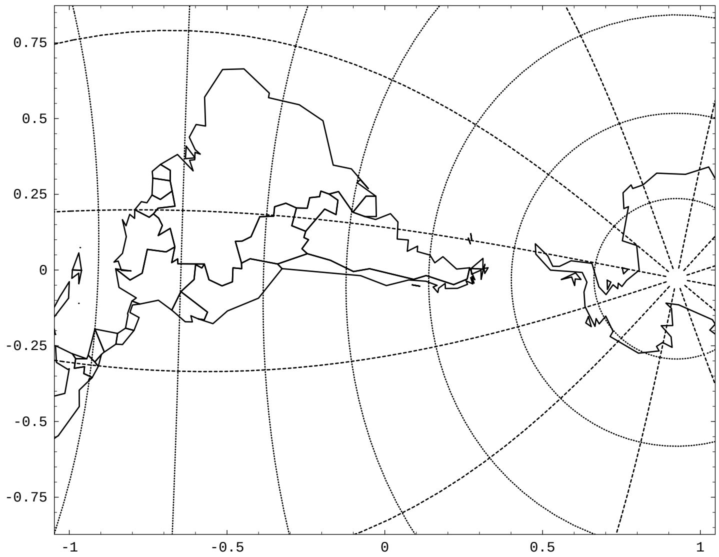

FIG. 4 – *A possible effect of tilting the sphere (by $\beta = 88°$ here) before projection. This strategy allows a minimization of the elongation of the domain along the y direction for elongated countries or areas of interest.*

$$\begin{aligned}
\theta'' &= \arcsin\left[\cos\beta\sin\theta' + \sin\beta\cos\theta'\sin\lambda'\right] \\
\cos\lambda''_{\text{prov}} &= \frac{1}{\cos\theta''}\left[\sin\beta\sin\theta' - \cos\beta\cos\theta'\sin\lambda'\right] \\
\sin\lambda''_{\text{prov}} &= \frac{1}{\cos\theta''}\left[\cos\theta'\cos\lambda'\right]
\end{aligned}$$

The subscript "prov" is to indicate that the result is not yet exactely the desired one, since this transformation brings the upper pole of the rotated sphere UP' on the origin meridian of the rotated/tilted sphere ($\lambda'' = 0$), although after the tilting it should be located at longitude $\lambda'' = -\pi/2$. As a consequence, a shift of $-\pi/2$ must be applied to λ''_{prov} in order to find the final value λ'' :

$$\lambda'' = \lambda''_{\text{prov}} - \pi/2$$

and we have :

$$\begin{aligned}
\cos\lambda'' &= \sin\lambda''_{\text{prov}} \\
\sin\lambda'' &= -\cos\lambda''_{\text{prov}}
\end{aligned}$$

The coordinate of the point (λ', θ') thus become after the tilting :

$$\theta'' = \arcsin\left[\cos\beta\sin\theta' + \sin\beta\cos\theta'\sin\lambda'\right] \quad (30)$$

$$\cos\lambda'' = \frac{1}{\cos\theta''}\left[\cos\theta'\cos\lambda'\right] \quad (31)$$

$$\sin\lambda'' = -\frac{1}{\cos\theta''}\left[\sin\beta\sin\theta' - \cos\beta\cos\theta'\sin\lambda'\right] \quad (32)$$

Case $\beta < 0$:

Applying (7)-(9) to (29), the coordinates (in the rotated/tilted geometry) of a point which has coordinates (λ', θ') in the rotated geometry will be, after the tilting :

$$\begin{aligned}
\theta'' &= \arcsin\left[\cos\beta\sin\theta' + \sin\beta\cos\theta'\sin\lambda'\right] \\
\cos\lambda''_{\text{prov}} &= \frac{1}{\cos\theta''}\left[-\sin\beta\sin\theta' + \cos\beta\cos\theta'\sin\lambda'\right] \\
\sin\lambda''_{\text{prov}} &= -\frac{1}{\cos\theta''}\left[\cos\theta'\cos\lambda'\right]
\end{aligned}$$

The subscript "prov" is to indicate that the result is not yet exactely the desired one, since this transformation brings the upper pole of the rotated sphere on the origin meridian of the rotated/tilted sphere ($\lambda'' = 0$), although after the tilting it should be located at longitude $\lambda'' = +\pi/2$. As a consequence, a shift of $\pi/2$ must be applied to λ''_{prov} in order to find the final value λ'' :

$$\lambda'' = \lambda''_{\text{prov}} + \pi/2$$

and we have :

$$\begin{aligned}
\cos\lambda'' &= -\sin\lambda''_{\text{prov}} \\
\sin\lambda'' &= \cos\lambda''_{\text{prov}}
\end{aligned}$$

The coordinate of the point (λ', θ') thus become after the tilting :

$$\begin{aligned}
\theta'' &= \arcsin\left[\cos\beta\sin\theta' + \sin\beta\cos\theta'\sin\lambda'\right] && (33) \\
\cos\lambda'' &= \frac{1}{\cos\theta''}\left[\cos\theta'\cos\lambda'\right] && (34) \\
\sin\lambda'' &= -\frac{1}{\cos\theta''}\left[\sin\beta\sin\theta' - \cos\beta\cos\theta'\sin\lambda'\right] && (35)
\end{aligned}$$

For both values of the sign of β, the expressions of (λ'', θ'') are finally identical. For $\beta = 0$ (which should in practice be the most usual case), the formulae also apply and provide $\lambda'' = \lambda'$ and $\theta'' = \theta'$.

Safeguard : *For $\theta'' = 0$ (that is for the two points which are brought to the two poles of the rotated/tilted sphere, formulae (31)–(32) have to be ignored, and the rotated longitude λ'' has to be defined conventionally (e.g. $\lambda'' = 0$). This conventional definition has of course no impact on the results of subsequent geometric transformations.*

It can be noticed that θ'' has a rather simple expression as a function of the geographical coordinates :

$$\sin\theta'' = \cos\beta\left[\cos\theta_0\sin\theta - \sin\theta_0\cos\theta\cos(\lambda - \lambda_0)\right] + \sin\beta\cos\theta\sin(\lambda - \lambda_0) \qquad (36)$$

This equation will be useful for the computation of the Northward direction.

2.8 Mercator projection

Now that we have a single formulation to describe the rotation which brings P_0 at the equator with the desired properties in longitude and tilting, we are ready for applying the classical Mercator projection to this rotated sphere.
Let $(x,\ y)$ be the coordinates in the projected plane of the point which has the geographical coordinates $(\lambda,\ \theta)$ and the coordinates $(\lambda'',\ \theta'')$ in the rotated/tilted geometry. Since the projection is the Mercator's one, we have :

$$
\begin{aligned}
x &= a\lambda'' && (37)\\
y &= -a\ln\left[\tan\left(\frac{\pi}{4}-\frac{\theta''}{2}\right)\right] \equiv a\ln\left[\tan\left(\frac{\pi}{4}+\frac{\theta''}{2}\right)\right] && (38)
\end{aligned}
$$

Therefore, we are able to compute the coordinates $(x,\ y)$ of any point on the sphere with geographical coordinates $(\lambda,\ \theta)$, except, of course for the two poles of the rotated/tilted sphere $(\cos\theta''=0)$.

Safeguard : *The LAM domain should be defined in such a way that it does not contains the poles of the rotated/tilted sphere. In other words the domain must not be unbounded in the y direction. In practice, for applications such as Aladin, this condition is fulfilled by nature since the size of the domain in the y direction is ultimately defined through a real machine-number, which by nature cannot be unbounded.*

The results of all this section are useful for plotting a map in the plane starting from geographical data, however, they do not allow to solve the meteorological equations in the plane since solving these equations rather needs the knowledge of the geographical position $(\lambda,\ \theta)$ for a given point $(x,\ y)$ in the plane, which is the reverse problem (see section 3 below).

3 reverse problem

In this section we seek the geographical coordinates $(\lambda,\ \theta)$ of a point which is defined by its Cartesian coordinates $(x,\ y)$.
Inverting (37)-(38) leads to :

$$
\begin{aligned}
\lambda'' &= \frac{x}{a} && (39)\\
\theta'' &= \frac{\pi}{2} - 2\arctan\left[\exp\left(-\frac{y}{a}\right)\right] && (40)
\end{aligned}
$$

This gives the rotated/tilted spherical coordinates $(\lambda'',\ \theta'')$ of the considered point. The reverse transform of the tilting has first to be applied. Hence we have to invert (30)-(32), which leads to :

$$
\begin{aligned}
\theta' &= \arcsin\left[\cos\beta\sin\theta'' - \sin\beta\cos\theta''\sin\lambda''\right] && (41)\\
\cos\lambda' &= \left(\frac{1}{\cos\theta'}\right)\left[\cos\theta''\cos\lambda''\right] && (42)\\
\sin\lambda' &= \left(\frac{1}{\cos\theta'}\right)\left[\sin\beta\sin\theta'' + \cos\beta\cos\theta''\sin\lambda''\right] && (43)
\end{aligned}
$$

The geographical coordinates are then obtained by directly inverting (10)-(12), which yields :

$$\theta = \arcsin\left[\cos\theta_0 \sin\theta' + \sin\theta_0 \cos\theta' \cos\lambda'\right] \tag{44}$$

$$\cos(\lambda - \lambda_0) = \left(\frac{1}{\cos\theta}\right)\left[-\sin\theta_0 \sin\theta' + \cos\theta_0 \cos\theta' \cos\lambda'\right] \tag{45}$$

$$\sin(\lambda - \lambda_0) = \left(\frac{1}{\cos\theta}\right)\left[\cos\theta' \sin\lambda'\right] \tag{46}$$

Safeguards : *Similar safeguards as for the direct transform have to be set, but now for* $\cos\theta' = 0$ *and* $\cos\theta = 0$ *: in these case the previous formulae for* λ' *and* λ *respectively have to be ignored and these quantities must be defined conventionally (still without consequences on the results of subsequent computations).*

4 map factor

4.1 Map factor in (λ'',θ'')

For a Mercator projection, the map factor m simply writes :

$$\mathsf{m} = 1/\cos\theta'' \tag{47}$$

4.2 Map factor in (x,y)

Starting from (40), we have :

$$\theta'' = \frac{\pi}{2} - 2\arctan\psi$$

with $\psi = \exp(-y/a)$. Hence $\mathsf{m} = \cos(\pi/2 - 2\arctan\psi) = \sin(2\arctan\psi)$. We use the 'half-tangent' formula :

$$\sin(2t) = \frac{2\tan t}{1 + \tan^2 t}$$

Substituting in (47) then leads, after some simple manipulations, to :

$$\mathsf{m} = \cosh\left(\frac{y}{a}\right) \tag{48}$$

5 Transformation for the vectors

In the transformed (rotated/tilted/projected) frame, we need to know the direction of the geographical North, which will denoted by the unit vector $\mathbf{j}$ in the local projected frame (the unit vector $\mathbf{i}$ is similarly defined as pointing towards the geographical eastward direction in this frame). We denote ($\mathbf{i}'$, $\mathbf{j}'$) the unit vectors of the local transformed frame in the rotated/tilted Mercator geometry (that is, respectively pointing toward $x > 0$ and $y > 0$). We write :

$$\begin{pmatrix} \mathbf{i} \\ \mathbf{j} \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \mathbf{i}' \\ \mathbf{j}' \end{pmatrix} \tag{49}$$

Consequently, α represents the angle between the direction of the geographical North **j** and and the ordinate axis **j'**. The angle α is referred to as "the compass" in the Aladin jargon. By construction of the above matrix (and in the Aladin code), α is thus counted positively clockwise as depicted in Fig.5.

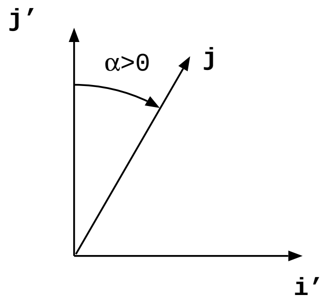

FIG. 5 – *Convention for the definition of α.*

It should be first noticed that the Mercator projection does not modify the angles with respect to the $y > 0$ axis. This is because the meridians are parallel to the $y > 0$ axis and the Mercator projection is a conformal one. As a consequence, α is equal to α', the angle between the Northward direction and the direction of meridians in the rotated/tilted (*but not projected*) geometry, in the local tangent-plane frame. In other words, Fig. 5 can be viewed as being also valid for the local tangent-plane in the rotated/tilted geometry. In this case $\mathbf{j}'$ is the unit vector pointing along the $\theta'' > 0$ direction, and $\mathbf{i}'$ is the unit vector pointing along the $\lambda'' > 0$ direction. In the following, we thus restrict ourselves to the computation of α' (instead of α), but due to the identity between α and α' we drop the prime of α'.

For any given point $P = (x, y)$ in the Cartesian domain, we can find its rotated/tilted coordinates (λ'', θ'') through (40)-(39), and its geographical coordinates (λ, θ) by the reverse transforms formulae (41)-(43) and (44)-(46).

There are several possibilities to compute α. The most straightforward conceptually would be to choose a point located just Northward of P, with a latitude shifted by $d\theta$, that is, $P_2 = (\lambda_2, \theta_2) = (\lambda, \theta + d\theta)$, and then to seek the Cartesian coordinates $(\lambda_2'', \theta_2'')$ of P_2 in the rotated/tilted geometry. this method works well and is conceptually simple, but leads to rather complicated expressions for α. A better method is presented in Clochard (1989). This method uses the fact that the two rotations involved to pass from (λ, θ) to (λ'', θ'') are always direct isometries. As a consequence, the geographical Northward direction is also always the direction which is directly orthogonal to the Eastward direction, in whatever of the rotated or rotated/tilted geometries. The idea is therefore to use the rather simple formula (36) and to differentiate this equation along the Northward and the Eastward directions respectively, for a unit-length differential element.

The length elements for Northward and Eastward small angular displacements are given by :

$$\delta_{\text{Northward}} = a d\theta \quad (50)$$

$$\delta_{\text{Eastward}} = a \cos\theta \, d\lambda \quad (51)$$

By differentiating (36) for $d\theta$ at constant λ, we find the corresponding displacement of θ'' in the rotated/tilted geometry, that we note $d\theta''_{d\theta}$:

$$\cos\theta'' d\theta''_{d\theta} = \{\cos\beta\,[\cos\theta_0\cos\theta + \sin\theta_0\sin\theta\cos(\lambda-\lambda_0)] - \sin\beta\sin\theta\sin(\lambda-\lambda_0)\}\,d\theta \tag{52}$$

The physical lengths displacements associated to the previous angular displacements are just given by multiplying the latter equation by a. A unit length displacement $ad\theta = 1$ corresponds to a displacement equal to $\mathbf{j}$, hence $a\theta''_{d\theta}$ also represent the component of $\mathbf{j}$ along $\mathbf{j}'$ in the frame $(\mathbf{i}', \mathbf{j}')$, that is, $\cos\alpha$. Finally, we have :

$$\cos\alpha = \left(\frac{1}{\cos\theta''}\right)\{\cos\beta\,[\cos\theta_0\cos\theta + \sin\theta_0\sin\theta\cos(\lambda-\lambda_0)] - \sin\beta\sin\theta\sin(\lambda-\lambda_0)\}$$

A similar method is employed to find $\sin\alpha$. By differentiating (36) for $d\lambda$ at constant θ, we find the corresponding displacement of θ'' in the rotated/tilted geometry, that we note $d\theta''_{d\lambda}$:

$$\cos\theta'' d\theta''_{d\lambda} = [\cos\beta\sin\theta_0\sin(\lambda-\lambda_0) + \sin\beta\cos(\lambda-\lambda_0)]\cos\theta d\lambda \tag{53}$$

The physical lengths displacements associated to the previous angular displacements are just given by multiplying the latter equation by a. A unit length displacement $a\cos\theta d\lambda = 1$ corresponds to a displacement equal to $\mathbf{i}$, hence $a\theta''_{d\lambda}$ also represent the component of $\mathbf{i}$ along $\mathbf{j}'$ in the frame $(\mathbf{i}', \mathbf{j}')$, that is, $(-\sin\alpha)$. Finally, we have :

$$\sin\alpha = -\left(\frac{1}{\cos\theta''}\right)[\cos\beta\sin\theta_0\sin(\lambda-\lambda_0) + \sin\beta\cos(\lambda-\lambda_0)]$$

To summarize, the trigonometric lines of the local "compass" are :

$$\cos\alpha = \left(\frac{1}{\cos\theta''}\right)\{\cos\beta\,[\cos\theta_0\cos\theta + \sin\theta_0\sin\theta\cos(\lambda-\lambda_0)] - \sin\beta\sin\theta\sin(\lambda-\lambda_0)\} \tag{54}$$

$$\sin\alpha = -\left(\frac{1}{\cos\theta''}\right)[\cos\beta\sin\theta_0\sin(\lambda-\lambda_0) + \sin\beta\cos(\lambda-\lambda_0)] \tag{55}$$

Beware that in the ARPEGE and Aladin codes, the trigonometric lines of α are stored as :

$$\begin{aligned} \texttt{GNORDL} &= \sin\alpha \\ \texttt{GNORDM} &= \cos\alpha \end{aligned}$$

We see that $(\texttt{GNORDL}, \texttt{GNORDM})$ are the components of the northward unit vector $\mathbf{j}$ in the $(\mathbf{i}', \mathbf{j}')$ frame. It should be noted that all these results are valid for both $\theta_0 \geq 0$ or $\theta_0 < 0$.

We have :

$$\begin{pmatrix} u_g \\ v_g \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \tag{56}$$

where (u_g, v_g) are the physical components of the wind in the geographical frame, and (u, v) are the physical components of the wind in the projected frame.

6 Coriolis parameter

The computation of the Coriolis factor $\sin\theta$ for a point of coordinates (x, y) is detailed here. The spherical coordinates (λ'', θ'') of the point in the rotated/tilted geometry are given by (39)-(40). Then the spherical coordinates (λ', θ') of this point in the rotated geometry are given by (43)-(41). Then, the Coriolis factor $\sin\theta$ is given by (44) :

$$\sin\theta = \cos\theta_0 \sin\theta' + \sin\theta_0 \cos\theta' \cos\lambda' \tag{57}$$

7 Use of the new geometry for meteorological equations

7.1 Projected meteorological equations

For a non-rotating planet, the general equations for the Fluid Mechanics on a sphere are valid for any system of spherical coordinates (λ, θ) (rotated or non-rotated) due to the invariance principle. The only potentially problematic term is the Coriolis term which appears for a rotating planet, since it refers to a fixed absolute direction and may therefore not be invariant through a change of spherical coordinates. However, the Coriolis force for the horizontal wind is just a rotation of the wind vector by $\pi/2$ and a multiplication by the local Coriolis parameter f. As a consequence, the Coriolis force expressed for the components of the wind in the rotated/tilted geometry are unchanged with respect to its form in the geographical geometry. In other words, the form of the Coriolis force is invariant by the transformation (56).
The projection of the equations brings some changes, that is, leads to the appearance of curvature terms. However, the curvature terms for a Mercator projection are already implemented in the Aladin code, and need not to be modified. In Aladin, the curvature terms appear only in the Eulerian version ; in the SL version they are absent. Anyway, this discussion does not seem to interfer with the proposed change of geometry.
Finally, the form of the meterorological equations in the rotated/tilted and projected frame is unchanged with respect to the projected equations already implemented in Aladin.

7.2 Semi-Lagrangian trajectories

For all the Aladin geometries, the SL trajectories are assumed to be straight lines in the projected plane (we do not search origin points on the great circle originating at the final point and directed toward the middle point wind). As a consequence, the research of origin points does not involve any spherical geometric computation. The only needed formulae are the reverse transform formulae, in order to compute the spherical coordinates of the origin point, once determined in the pure Cartesian LAM frame (these spherical coordinates are needed namely for computing the physics at the origin point).

8 Other points

8.1 Winds to work with

Throughout almost all the integration model (except sompe particular parts of the physics), the computations are performed in the projected geometry. This means that all the dynamical equations are cast as function of x and y, and that the wind used in the model is always the wind in the frame of the projected geometry (u, v). Two variants are used for this wind in the projected geometry :

- Physical wind in the frame of the projected geometry $\mathbf{V} = (u, v)$. This variant is used in all the grid-point computations.
- "Wind on the map" in the frame of the projected geometry $\mathbf{V}' = (u', v')$ (also called "reduced wind"). This variant is used in all spectral computations.

Since $\mathbf{V} = \mathsf{m}\mathbf{V}'$, the transformation between these two types of wind is straightforward, and needs not to be changed in the Aladin code for the implementation of the new geometry.

8.2 Coordinates to be stored

The parts of the model which use a description in the geographical frame or geographic winds are restricted to "peripheric" tasks like the post-processing or the comparison with observations (and of course also the geometry package). For these parts, the knowledge of the relationship between the Cartesian LAM frame and the geographical frame is sufficient for any purpose, and they use these transformations formulae in a transparent way. As a consequence, in all the LAM system, only the transformation $(x, y) \rightarrow (\lambda, \theta)$ needs to be stored in the setup part of the model ; that is, we need to store only (λ, θ) on the grid, and we never need to store (λ'', θ''). This features is thus compatible with the existing code, in which only one set of spherical coordinates is stored.

8.3 Coriolis terms in 4D-VAR

The Coriolis parameter f is used in the balance equation for the computation of the background error cost function J_B in the 4D-VAR configuration. For this purpose (as for the dynamics) the Coriolis parameter is directly read from the array `RCORI`, computed in the setup and the form of the balance equation as a function of f is unchanged compared to the other existing geometries. As a consequence, it is sufficient to modify the filling of the array `RCORI`, and no modification of the form of the Coriolis terms in the model equations is needed.

Remark : The balance equation being non-linear, it is computed in spectral space through a spectral product with a low-order fitted value of the Coriolis parameter (currently with a $N = 15$ spectral truncation). However, for sake of simplicity, the current fit it performed along the y direction only, in an hard-coded manner. This limitation is of course not well-adapted to the case of a tilted geometry, for which the iso-f curves are tilted by construction. In the case of the Fig. 4 for instance, the only variations of f are along the x axis, and the spectral fit along y would therefore give nothing better than a fit by a constant ! Moreover, this limitation of the spectral fit of f for the balance equation of J_B is also already present in the current Lambert geometry, which allows a tilting when $\lambda_0 \neq \lambda_c$ as in the domain depicted in the right part of Fig. 6. When the Lambert geometry is tilted, the iso-f can be strongly tilted with respect to the y axis, and the abovementionned limitation also acts in a similar manner.
This discussion shows that in the current state,the balance equation of the J_B cannot be used with strongly tilted geometries (tilted/Lambert or rotated/tilted Mercator). This seems to plead more for a revision of the fitting algorithm of f in J_B than for a restriction of the tilting of LAM geometries. This point is not urgent but must be kept in mind for the day we will want to use a strongly tilted geometry together with a 4D-VAR LAM.

8.4 Full-Pos

The change of geometry should be transparent for the post-processing Full-Pos, provided the "compass" arrays are correctly filled.

8.5 File handling

Provided that the ascending compatibility is guaranteed, and that the "frame/header" (the so-called "cadre" in french) of the files is not modified, this should not rise any specific problem.

9 Comparison with existing geometries (EGGX and EGGPACK packages)

The old `EGGX` package originally contained an optional rotation of the area of interest to the equator of the rotated sphere, and the possibility of having a Mercator projection (the possibility of a tilting was not implemented for the rotated Mercator projection in `EGGX`), as proposed here. However, this package also retained the possibility for various geometries (e.g. rotated lat-lon on the sphere, rotated stereo-Lambert) which made it rather complicated. Here, only the simplest case of projected rotated geometry (the Mercator one) is included, leading to a rather simple problem.
The more recent `EGGPACK` package has eliminated the possibility of applying a rotation of the area of interest to the Equator before the projection, and thus restricts itself to conventional Stero-Lambert and Mercator projections. The type of projection is automatically deduced from the latitude of the reference point of the projection.

The major difference between the `EGGX` and `EGGPACK` packages comes however from the nature of the free parameters. In `EGGX`, many options were implemented :

- the type of projection (or spherical lat-lon) can be imposed, or automatically chosen to minimize the variations of the map factor in the domain.
- the (integer) dimensions of the grid-arrays are free paremeters
- at least the SW corner is imposed as a free parameter
- the NE corner can be fully specified, or modified in order to fulfil a constraint of isotropic resolution.
- the reference point of the projection can be specified, or automatically computed

This strategy was very flexible and allows many types of projection/domain specifications, but is, on the other hand, quite complicated. The aim of the `EGGPACK` package was to eliminate the less used capabilities and to allow a more intuitive specification of the free parameters.
In `EGGPACK`, the strategy for the specification of free parameters is as follows :

- the reference point $P_0 = (\lambda_0, \theta_0)$ for the projection is provided.
- the type of projection is automatically chosen to have a projection tangent at the reference point (moreover, spherical lat-lon and rotated projections are eliminated).
- the (integer) dimensions of the grid-arrays are free paremeters
- the center of the LAM domain $P_c = (\lambda_c, \theta_c)$ is specified
- the (uniform) resolutions in x and y are specified.

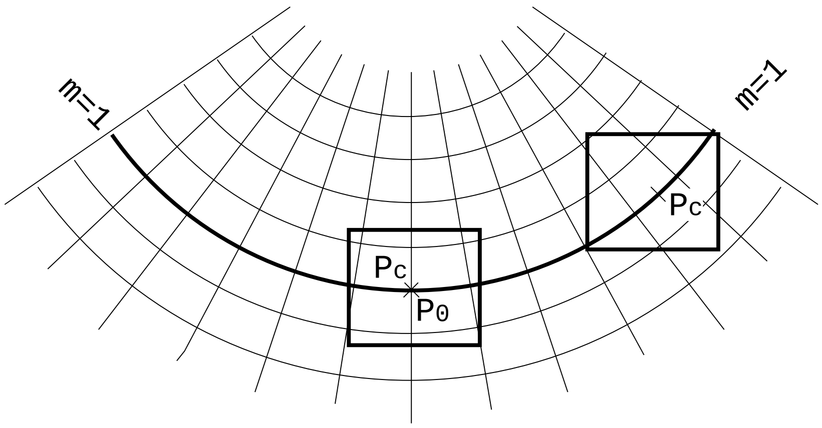

FIG. 6 – *Effect of having $P_c = P_0$ (domain at centre) and $P_c \neq P_0$ (domain at right) in the case of a Lambert projection.*

It should be noted that the reference point P_0 of the projection (the place where $\mathsf{m} = 1$ and the geographical North is along the y direction) can be taken not at the center of the domain P_c in `EGGPACK`. Chosing P_c outside of the line $\mathsf{m} = 1$ would be detrimental since in this case, the variations of the map factor ar not minimized in the LAM domain. Hence we assume that whenever different from P_0, P_c always lies on the line $\mathsf{m} = 1$ in this discussion, and therefore, the only degree of freedom is the longitude λ_c of the domain's center (i.e. θ_c must be equal to θ_0). This degree of freedom $\lambda_c \neq \lambda_0$ results, in the case of a Lambert projection, in a "tilting" effect similar to the one obtained through β presented in this document : the circle $\mathsf{m} = 1$ is tilted with respect to the case where $P_0 = P_c$ (see figure 6).

A similar method could be used for obtaining a tilting effect in the rotated Mercator projection. In this case we would choose a reference point P_0 for the projection, and choose the center of the domain P_c elsewhere. Since choosing P_c away from the equator of the rotated geometry would be detrimental for the map factor, we have to choose P_c in such a way that $\theta'_c = 0$, i.e. λ'_c is the only degree of freedom. Similarly to the conventional Lambert projection, this determines a unique degree of freedom for the couple (λ_c, θ_c), which has to travel only on a great circle on the earth. Therefore, this degree of freedom could be restricted to the choice of λ_c (with the disadvantage of needing some special case when the latter curve is a meridional great circle, as it occurs if $\cos(\theta_0) = 0$).

However this methods possesses the disadvantage of being not very intuitive : a trial and error process is most often necessary in order to obtain a given desired domain. This process can be avoided by writing a small program which computes the mysterious needed value of λ_c as a function of some specified parameters, but the interest of such a complication is questionable if a more intuitive and simple approach is possible. Moreover, choosing P_c far away from P_0 may result in undesired complications if the domain is large : in this case the domain may no longer be a connex piece of the original "plane ribbon", because it goes beyond the lateral edges of the

global projected domain. Then special safeguards would probably have to be installed in order to maintain the operativeness of the computations.
This is why in this document, the tilting is directly applied before the projection, and in the form of a pure spherical tilting rotation , in order to allow a much more intuitive actual tilting effect for the rotated Mercator projection. In this case, the distinction between P_0 and P_c is no longer needed, and therefore P_c is not a free parameter, the only one being P_0.

The possibility of a rotated/tilted Mercator geometry can therefore be inserted in the new `EGGPACK` package, provided that the degree of freedom $P_c \neq P_0$ is replaced by the specification of β. The free parameter λ_c must be replaced by (or acquire the signification of) β.

10 Coding and user aspects

10.1 Coding aspects

The FORTAN code related to this new geometry exactly follows the computations presented in section 2–6, except some very specific points which are mostly due to the application to finite-precision computers. The differences between the exact formulae in 2–6 and the practical code are listed below :

- The Arcsin and Arcos function have needed to be armoured against arguments with magnitude larger than one. Formally these functions must therefore been thought as e.g.

$$\arcsin(x) = \arcsin(\max[\min[x, 1], -1]) \tag{58}$$

This modification was required since in some cases, the finite-precision magnitude of the argument was experienced to very slightly exceed one.

10.2 User aspects

After some discussions inside the group, choice was made to implement the tilting functionality through a direct tilting angle β instead of an approach using a "tilting-like" longitude, because it leads to a more friendly and intuitive definition of the geometry and domain (see also discussion in section 9 above). In the case where $\theta_0 = \pm 90°$, it is important to note that both λ_0 and β continue to play their normal role, although the role of these two free-parameters is totally redundant in this case. This redundancy was left in order to allow a continuity of the domain variations when the value $\theta_0 = \pm 90°$ is reached continuously.

From the point of view of the user, a call of the new geometry functionality is needed only the case where the one of the three following configurations is used :
- E923 (making a LAM climatology file)
- E927 (making a LAM initial/coupling file from a global historical file)
- EE927 (making a LAM initial/coupling file from a bigger LAM historical file)

For a forecast (configuration 001), the geometry is read in the "cadre" of the initial file and hence the geometry of the domain does not need to be re-specified.

The new geometry is inserted as a new functionality of the existing code `EGGPACK`. Hence the activation of the new geometry is done through a switch in the existing package. The rotated/tilted geometry is conventionally activated by setting `LMRT=.TRUE.` in the namelist `NEMGEO`. This has

to consequence to set the value of the "cadre" parameter NROTEQ to -2 (the value NROTEQ $= -1$ is for nonrotated Stereo-Lambert and Mercator projections, while positive values are used for even older geometries).
Here are the name of the free-parameters to specify in the namelist NEMGEO for obtaining a given geometry and domain :

LMRT : activation or not of the Mercator Rotated/Tilted geometry

ELONC : longitude of the center of the domain (λ_0 in the present documentation).

ELATC : latitude of the center of the domain (θ_0 in the present documentation)..

ELON0 : tilting angle (β in the present documentation).

EDELX : resolution (in meters) on the map in the x direction.

EDELY : resolution (in meters) on the map in the y direction.

Notes :

- All angles are to be specified in degrees

- Although the tilting angle β is specified in a free-parameter which looks like a "reference longitude", this name is purely conventional and β actually has the meaning which has been defined in the present paper, as discussed in the first paragraph of this sub-section.

- The parameter ELAT0 in the namelist NEMGEO is ignored for the rotated/tilted Mercator geometry.

- By construction the resolution taken on the map is uniform in a given direction x or y (it is however non uniform in the geographical space, since increasing from the rotated/tilted equator toward the rotated/tilted poles). In practice, the uniform resolution to be specified is the geographical resolution which is wished at the center of the domain, since these two concepts coincide there, because the map factor is equal to one.

11 Conclusions

A rotated/tilted Mercator geometry for Aladin has been described :

- The fields derived from the geometry itself can be quite easily computed, even if a bit more trigonometry is needed compared to the standard Mercator projection. This step is done in the setup part of the model, and the change of geometry should not result in overcosts in the time-loop or in the 4D-VAR.
- The new geometry is inserted as a new functionality of the existing code EGGPACK.
- Finally, the meteorological dynamical equations in the Cartesian LAM plane do not need to be substantially modified.