Tomaž Vrhovec
University Ljubljana
Dept. Mathematics and Physics

Selected lectures on numerical analysis in NWP

Presented in Kranjska gora, ALATNET, May 27 th an 28 th 2002

1. Grid point presentation of meteorological variables

Meteorological variables (u,v,w, T, p, etc) are: 3D, nonstationary, continuous and have continuous derivatives, defined in all points of space/time

Grid point representation – finite differences representation:
The variables are known just at discrete point (in a gird points, mesh points)
- sampling (measurement) is made in points
- model variables represent space and time mean over grid box and time interval
- the equations have to be adjusted to this averaging

$$\overline{\omega} = \frac{1}{\Delta x \Delta y \Delta z \Delta t} \int_{o}^{\Delta x \Delta y \Delta z \Delta t} \omega dxdydzdt$$

$$\omega = \overline{\omega} + \omega'$$

$$\overline{\omega'} = 0$$
$$\overline{\overline{\omega}} = \overline{\omega}$$

For example: we use linear advection equation.

$$\frac{\partial \omega}{\partial t} = -c \frac{\partial \omega}{\partial x}$$

All the variables are split into mean and subgrid parts.

Mean of sub grid part is 0. When inserting the two parts into equation and averaging the equation over grid box the equation for mean values is derived. If equation is linear, mean values time derivatives are dependant on means only, if nonlinear the product terms of perturbations stay in the equation.

$$\frac{\partial \overline{\omega}}{\partial t} = -c \frac{\partial \overline{\omega}}{\partial x}$$

When calculating derivatives mean values from neighboring grid boxes are used. The derivatives approximation are valid if grid size goes towards 0

$$\Delta x, \Delta y, \Delta z, \Delta t \to 0$$

and if the algebraic differential form limits towards analytical expression of derivative.

$$\frac{\partial \omega}{\partial t} \Big|_j \approx \frac{\omega_j^{n+1} - \omega_n^n}{\Delta t}$$

$$\frac{\partial \omega}{\partial x} \Big|^n \approx \frac{\omega_{j+1}^n - \omega_{jn}^n}{\Delta x}$$

2. Spectral representation of meteorological variables

Variables can be expressed as an infinite sum of time dependant weights multiplied by spatial functions. In numerical representation the sum is finite (truncation).

$$\hat{\omega} = \sum_{m=0}^{M} \omega_m(t)\psi_m$$

$$\psi_m(x,y,z)$$

Spatial functions are eigenfunctions of linear differential operator upon the area of interest (boundary dependant) and orthogonal on the domain.

$$\frac{\partial(L(\omega))}{\partial t} = F(\omega)$$

$$L(\psi_m) + \varepsilon_m \psi_m = 0$$

In linear advection case L is simple and eigen functions are waves on the sphere or on the plain, also other orthogonal functions can be used (chapeau, delta – finite elements methods)

Meteorological equations are to be written in form for finite sums. All the truncation error is located at functions that have their wave numbers above the maximum M^{max}.

Example is again linear advection equation, this time in cyclic zonal domain.

$$\hat{\omega}(\lambda,t) = \sum_{m=-M}^{M} \omega_m(t)\exp(im\lambda)$$

Only the wave components with numbers under M^{max} are included in calculation of time derivatives.
Because of orthogonality the system of algebraic equation is a set of independent eqs. for the weights

$$\frac{\partial \omega_m}{\partial t} = -c\frac{\partial \omega_m}{\partial \lambda}$$

With nonlinear adv.eq. system gets more complicated but still error stays with waves with wavenumber over M^{max}

Space derivatives are analytical functions

$$\frac{\partial \widehat{\omega}}{\partial \lambda} = \sum_{m=0}^{M} \omega_m(t) \frac{\partial \psi_m}{\partial \lambda}$$

while time derivatives of weights are still expressed in finite difference form.

Transformation from weights into complete function is inverse Fourier transformation, from function into weights

$$\omega_m = \int_0^{2\Pi} \widehat{\omega} \exp(-im\lambda) d\lambda$$

3. Grid structure

Horizontal grids

Staggering due to separation of solutions for momentum and geopotential => noise at the shortest waves (two Δx wave is often stationary)

Example: 1d gravity waves:

$$\frac{\partial u}{\partial t} = -g\frac{\partial h}{\partial x}$$
$$\frac{\partial h}{\partial t} = -H\frac{\partial u}{\partial x}$$

phase speed:
$$c_f = \sqrt{gH}$$

numerical phase speed
$$c_f^{\ *} = \sqrt{gH}\,\frac{\sin k\Delta x}{k\Delta x}$$

Solution: staggering: geopotential and momentum are not calculated in the same points but one in between other.

Vertical grids

In terrain following coordinates (σ) the lowest coordinate surface coincides with topography. Vectors (wind velocity) has to be transformed in contavariant sense, also equations have to be transformed accordingly into contravariant format (additional terms in the eqs of motion).

Boundary condition at the bottom is uniform in all model domain (always at the same computational level) and vertical (3rd component, contavariant) of wind is always orthogonal to the topography so is simply 0.

Staggering is a *rule*, so that geopotential and horizontal momentum are not at the ground level.

Stretching

Nonuniform distances between grid points (in any of directions).

Vertical stretching is a rule, as PBL has to be better resolved than free atmosphere.

$\Delta\sigma = f(p)$

Horizontal stretching

a) choice of coordinate system: if spherical then Cartesian distances between the grid points shrink when approaching spherical poles. (Rotation of spherical coordinate system so that its pole is dislocated from geographical pole)
b) stretching of $\Delta\varphi$ so that angular distances between grid point in antipodes are greater then around the area of interest
c) stretching of both horizontal coordinates: used in limited area or mesoscale models to put boundaries as far away from the area of interest (problems with stability (CFL), problems with waves refraction..)

Stretching: complicates definitions of grid points averages and finite difference spatial derivatives in grid point representation but does not complicate calculation of spatial derivatives in spectral

<u>Nesting</u>

a) to get fine scale resolution in the area of interest (AOI)
b) to employ different physical packages in different resolutions as in better resolutions additional physical processes have to be deparametrisized
c) to economize calculation (few global – many local models)
d) to avoid detailed calculation in the areas of low initial data density
e) to get lateral boundary conditions for next nest (reduction of grid dimensions should be gradual)
f) each nest must be framed by a transition cone
g) nest can move with a process

One way nesting: boundary conditions from larger scale to smaller: consecutive calculation, large scale first, then (also possible with delay) smaller scale model. BC update interval.

Two way nesting: boundary conditions are exchanged in both ways. Smaller scale model performs more time steps and can use also different physics – usually slower calculation in smaller scale. Large scale model has to wait for smaller scale results. BC update interval is one large scale time step.

4. **Boundary conditions**

Well posed problems – for each variable and its each derivative boundary condition in each dimension

<u>Lateral BC</u>

Global model has no lateral boundaries.
Nesting
Boundary as far as possible from the AOI
Mathematical simplifications:

- cyclic
- fixed values of variable (also updated in time)
- fixed values of variable spatial derivative
- combinations (sponge)

<u>Lower BC</u>

The only physical boundary of the atmosphere. Normal component of wind velocity vector is 0 at lower boundary. (contra variant form of terrain following coordinates) Temperature should be continuous towards the temperature of terrain surface: energy balance. Importance of good topography and phisiography increases with improvement of spatial resolution.

<u>Upper BC</u>

<u>Where is it?</u>

Vertical component of wind velocity should be 0 there

Above the stratosphere (as there are strong gravity waves there due to stratosphere stability)
Mathematical simplifications:

- rigid lid (reflects gravity waves)
- open lid – free boundary
- damping – gradual decrease of vertical velocity
- radiational – just some wavelengths are damped, others are refracted – Fourier transform in vertical direction

5. **Time integration**

Oscillation equation can be used as a test for various time integration schemes as gravity waves eq., and advection eq. can be transformed into oscillation equation. U(t)!

$$\frac{dU}{dt} = i\omega U$$

it is analytically solved by any function in form

$$U = U_0 \exp(i\omega t)$$

To transform linear advection equation into oscillation equation general solution of lin adv eq U = U(x-ct) must be particularized using traveling waves. The frequency is then

$$\omega = ck$$

To test numerical time integration schemes analytical solution is used:

$$U(t) = U_0 \exp(i\omega t)$$
$$U(n\Delta t) = U_0 \exp(i\omega(n\Delta t))$$
$$U((n+1)\Delta t) = U(n\Delta t) \ \ \exp(i\omega\Delta t)$$

$U((n+1)\Delta t)/U(n\Delta t) = \lambda$ - amplification factor

$$U((n+1)\Delta t) = \lambda^n U_o$$

For stability of a scheme

$$\lambda = \begin{cases} >1 ... unstable \\ =1 ... neutral \\ <1 ... stable, damping \end{cases}$$ good is neutral or some damping!

<u>2. level schemes</u>

<u>n, and n+1</u>

$(U^{n+1} - U^n)/\Delta t = F^n$ explicit

$(U^{n+1} - U^n)/\Delta t = F^{n+1}$ implicit

$(U^{n+1} - U^n)/\Delta t = F_1^{\ n} + F_2^{\ n+1} ... F_1 + F_2 = F$ semi implicit

$(U^{n+1} - U^n)/\Delta t = aF^n + (1-a)F^{n+1} ... a < 1$ trapezoid

iterative

<u>3. level schemes</u>
<u>n-1, n, n+1</u>

central:
leapfrog

$$\frac{dU}{dt} \approx (U^{n+1} - U^{n-1})/2\Delta t = F^n$$

Adams Bashfort:

$$\frac{dU}{dt} \approx (U^{n+1} - U^n)/\Delta t = 3/2F^n - 1/2F^{n-1}$$

noncentral: forward:

$$\frac{dU}{dt} \approx \frac{U^{n-1} - 4U^{n-1} + 3U^{n+}}{2\Delta t}$$

<u>4. level schemes</u>
<u>n-2, n-1, n, n+1</u>

forward

$$\frac{dU}{dt} \approx \frac{U^{n-2} - 6U^{n-1} + 3U^n + 2U^{n+1}}{6\Delta t}$$

5 <u>level scheme</u>

$$\frac{dU}{dt} \approx \frac{U^{n-2} - 8U^{n-1} + 8U^{n+1} + U^{n+2}}{12\Delta t}$$

All the schemes can be derived out of Taylor expansion series: higher accuracy as additional derivatives are expressed analytically. The same is for spatial derivatives.

Stability analysis for Leapfrog scheme explains CFL stability conditions (for explicit schemes)

6. Nonlinear advection equation

$$\frac{\partial \omega}{\partial t} = -\omega \frac{\partial \omega}{\partial x}$$

is solved by

$$\omega = f(x - \omega t)$$

implicit solution

It is not solved by traveling wave

$$\omega = \omega_o \exp(i(kx - ct))$$

If we put a wave into nonlinear term we get double frequency, or double wave number solutions. Nonlinear equation already in its analytical form changes spectrum of waves.

In numerical world wave numbers are limited by $k_{max.}$

Shorter waves are in grid point representation recognized as longer ones

If $k > k_{max}$ then $k_{rec} = 2\, k_{max} - k$

ALLIASIN error

Energy is accumulating near kmax! The shortest wave can be stationary and increasing with time: increasing noise.

Grid point solutions:
- dispersive shames
- spectral damping
- filtering
- Arakawa jacobian

Spectral:
- with cutoff error also alliasing error is included

7. Eulerean – Lagrangian – Semilagrangian

1D:

$\omega(x,t)$

EULERIAN

$$\frac{\partial \omega}{\partial t} = -u\frac{\partial \omega}{\partial x} + F$$

- fixed points in space
- fixed advection scheme
- variable advection scheme: forward upstream

Lagrange:

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} = \frac{d\omega}{dt} = F$$

Determining F along a trajectory. Trajectories starting from initial condition. Can get complicated if trajectories converge, rotate, nonlinear advection.

Semilagrange:

Solving Lagrange equation but just in one time step. Final positions of trajectories within 1 time step are in a regular grid, it is necessary to find initial positions at the beginning of time interval.

8. Implications of basic atmospheric equations

Anelastic
Deep – shallow continuity equation
Bousinesq
Hydrostatic-nonhydrostatic