Extension of the equations for a moist precipitating atmosphere to mass-based coordinates in various contexts

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1 Introduction

This paper derives the equations for a moist precipitating atmosphere in mass coordinate. The starting point is the paper by Malardel on the equations for a precipitating atmosphere in height coordinate. Therefore basically, this paper deals with the coordinate change from height to mass coordinates for a moist precipitating atmosphere in the framework adopted therein. The technique for the coordinate change is largely taken from laprise (1992, MWR).

It should be outlined that the definition of the mass coordinate itself in a given model is very closely related to the choice of the density appearing in the continuity equation and in the Laplace equation actually used in this model. Therefore, unlike Malardel's paper, which delt with "systems" without necessarily having to refer explicitly to practical applications, this paper rather deals with "models" because for a given general system, various choices can be made concerning the actual density appearing in a corresponding model.

2 Change to a general vertical coordinate $z \rightarrow s(x, y, z)$ in a general context

In this section we derive the equations for the change of a vertical coordinate in the most general context useful for subsequent purposes. A general vertical coordinate s is introduced (whose only requirement being to remain monotonous with respect to z)

Let $s(x, y, z)$ be an arbitrary vertical coordinate:

$$\forall(z, s), \qquad \left.\frac{\partial s}{\partial z}\right|_{(x,y)} \neq 0 \tag{1}$$

$$\forall(z, s), \qquad \left.\frac{\partial z}{\partial s}\right|_{(x,y)} \neq 0 \tag{2}$$

Let $(d/dt_\bullet)$ be defined by:

$$\frac{d}{dt_\bullet} = \frac{\partial}{\partial t} + u_\bullet \left.\frac{\partial}{\partial x}\right|_z + v_\bullet \left.\frac{\partial}{\partial y}\right|_z + w_\bullet \left.\frac{\partial}{\partial z}\right|_x, \tag{3}$$

where $\mathbf{v}_\bullet = (u_\bullet, v_\bullet, w_\bullet)$ are velocities along (x, y, z) left unspecified for time being (velocity of gaseous component, or barycentric, or any other choice...). This velocity is the one used in the dynamical part (Lagrangian velocity, momentum). A more explicit specification for this velocity will be performed afterward, depending on the context for each examined case.

Let define $(\dot{s})_\bullet := (ds/dt_\bullet)$. Symmetrically to (3), we may also write:

$$\frac{d}{dt_\bullet} = \frac{\partial}{\partial t} + u_\bullet \left.\frac{\partial}{\partial x}\right|_s + v_\bullet \left.\frac{\partial}{\partial y}\right|_s + (\dot{s})_\bullet \left.\frac{\partial}{\partial s}\right|_{(x,y)}, \tag{4}$$

We have:

$$\left.\frac{\partial u_\bullet}{\partial x}\right|_z = \left.\frac{\partial u_\bullet}{\partial x}\right|_s + \left.\frac{\partial u_\bullet}{\partial s}\right|_{(x,y)} \left.\frac{\partial s}{\partial x}\right|_z \tag{5}$$

$$\left.\frac{\partial v_\bullet}{\partial y}\right|_z = \left.\frac{\partial v_\bullet}{\partial y}\right|_s + \left.\frac{\partial v_\bullet}{\partial s}\right|_{(x,y)} \left.\frac{\partial s}{\partial y}\right|_z \tag{6}$$

$$\left.\frac{\partial w_\bullet}{\partial z}\right|_{(x,y)} = \left.\frac{\partial w_\bullet}{\partial s}\right|_{(x,y)} \left.\frac{\partial s}{\partial z}\right|_{(x,y)} = \left.\frac{\partial s}{\partial z}\right|_{(x,y)} \frac{\partial}{\partial s}\left[\frac{dz}{dt_\bullet}\right]_{(x,y)} \tag{7}$$

Moreover, the cyclic relation yields:

$$\left.\frac{\partial \alpha}{\partial \beta}\right|_\gamma = -\left.\frac{\partial \alpha}{\partial \gamma}\right|_\beta \left.\frac{\partial \gamma}{\partial \beta}\right|_\alpha, \tag{8}$$

which implies:

$$\left.\frac{\partial w_\bullet}{\partial z}\right|_{(x,y)} = \left.\frac{\partial s}{\partial z}\right|_{(x,y)} \frac{d}{dt_\bullet}\left[\frac{\partial z}{\partial s}\right]_{(x,y)} - \left.\frac{\partial u_\bullet}{\partial s}\right|_{(x,y)} \left.\frac{\partial s}{\partial x}\right|_z - \left.\frac{\partial v_\bullet}{\partial s}\right|_{(x,y)} \left.\frac{\partial s}{\partial y}\right|_z + \left.\frac{\partial (\dot{s})_\bullet}{\partial s}\right|_{(x,y)}. \tag{9}$$

Finally, the numerical value for the 3D divergence $D_{3\bullet}$ of the velocity field is given by:

$$\begin{aligned} D_{3\bullet} &= \left.\frac{\partial u_\bullet}{\partial x}\right|_z + \left.\frac{\partial v_\bullet}{\partial y}\right|_z + \left.\frac{\partial w_\bullet}{\partial z}\right|_{(x,y)} \\ &\equiv \left.\frac{\partial u_\bullet}{\partial x}\right|_s + \left.\frac{\partial v_\bullet}{\partial y}\right|_s + \left.\frac{\partial (\dot{s})_\bullet}{\partial s}\right|_{(x,y)} + \left.\frac{\partial s}{\partial z}\right|_{(x,y)} \frac{d}{dt_\bullet} \left.\frac{\partial z}{\partial s}\right|_{(x,y)} \end{aligned} \tag{10}$$

This latter equation is a *mathematical* identity which does not contain any physics inside.

Note: *In all this paper, it is assumed that the velocity* $\mathbf{v}_\bullet$ *used in the dynamics (momentum, Lagrangian velocity) is necessarily the mass-averaged velocity of the matter considered in the continuity equation. This apparent loss of generality does not cause problems because this is actually the case for all applications considered here.*

Let the continuity equation of the atmosphere write in a general form:

$$\frac{d\rho_\bullet}{dt_\bullet} + \rho_\bullet D_{3\bullet} = \dot{R}_\bullet, \tag{11}$$

where, as outlined above, $\rho_\bullet$ is the density of the matter which mass-averaged velocity is $\mathbf{v}_\bullet$, and $\dot{R}_\bullet$ is a source term for $\rho_\bullet$. This source term in (11) may be identically vanishing or not, depending on the choices made for the definition of $\rho_\bullet$ and $(u_\bullet, v_\bullet, w_\bullet)$. For instance, as shown in Malardel's paper, $\dot{R}_\bullet$ vanishes if $\rho_\bullet$ and $\mathbf{v}_\bullet$ definitions are chosen a follow:

$$\rho_\bullet := \rho_a \quad \text{and} \quad (u_\bullet, v_\bullet, w_\bullet) := (u_a, v_a, w_a) \tag{12}$$

$$\rho_\bullet := \rho \quad \text{and} \quad (u_\bullet, v_\bullet, w_\bullet) := (u, v, w) \tag{13}$$

where Malardel's notations for the total density ρ and the total barycentric velocity (u, v, w) have been adopted. From now, the subscript $|_{(x,y)}$ appearing in any vertical derivative is dropped for conciseness.
The continuity equation rewrites as:

$$\frac{\partial \rho_\bullet}{\partial t} + u_\bullet \left.\frac{\partial \rho_\bullet}{\partial x}\right|_s + v_\bullet \left.\frac{\partial \rho_\bullet}{\partial y}\right|_s + (\dot{s})_\bullet \frac{\partial \rho_\bullet}{\partial s} + \rho_\bullet \left[\left.\frac{\partial u_\bullet}{\partial x}\right|_s + \left.\frac{\partial v_\bullet}{\partial y}\right|_s + \frac{\partial (\dot{s})_\bullet}{\partial s} + \frac{\partial s}{\partial z} \frac{d}{dt_\bullet} \frac{\partial z}{\partial s} \right] = \dot{R}_\bullet, \tag{14}$$

Multiplying by the non-vanishing and non-unbounded quantity $\frac{\partial z}{\partial s}$ yields:

$$
\frac{\partial z}{\partial s}\left[\frac{\partial \rho_\bullet}{\partial t}+u_\bullet\left.\frac{\partial \rho_\bullet}{\partial x}\right|_s+v_\bullet\left.\frac{\partial \rho_\bullet}{\partial y}\right|_s+(\dot{s})_\bullet\frac{\partial \rho_\bullet}{\partial s}+\rho_\bullet\left.\frac{\partial u_\bullet}{\partial x}\right|_s+\rho_\bullet\left.\frac{\partial v_\bullet}{\partial y}\right|_s+\rho_\bullet\frac{\partial (\dot{s})_\bullet}{\partial s}\right]
$$

$$
+\quad \rho_\bullet\left\{\frac{\partial}{\partial t}\frac{\partial z}{\partial s}+u_\bullet\left[\frac{\partial}{\partial x}\frac{\partial z}{\partial s}\right]_s+v_\bullet\left[\frac{\partial}{\partial y}\frac{\partial z}{\partial s}\right]_s+(\dot{s})_\bullet\left[\frac{\partial}{\partial s}\frac{\partial z}{\partial s}\right]\right\}=\frac{\partial z}{\partial s}\dot{R}_\bullet
$$

which finally writes:

$$
\frac{\partial}{\partial t}\left[\rho_\bullet\frac{\partial z}{\partial s}\right]+\frac{\partial}{\partial x}\left[\rho_\bullet u_\bullet\frac{\partial z}{\partial s}\right]_s+\frac{\partial}{\partial y}\left[\rho_\bullet v_\bullet\frac{\partial z}{\partial s}\right]_s+\frac{\partial}{\partial s}\left[\rho_\bullet(\dot{s})_\bullet\frac{\partial z}{\partial s}\right]=\frac{\partial z}{\partial s}\dot{R}_\bullet \tag{15}
$$

3 General mass-based coordinates

It is seen from (15) that choosing s such as $\frac{\partial}{\partial t}\left[\rho_\bullet\frac{\partial z}{\partial s}\right]\equiv 0$, leads to a diagnostic continuity equation in s coordinate. This class of coordinates is therefore the coordinates s for which $\rho_\bullet\frac{\partial z}{\partial s}$ is a pure constant. In the meteorological tradition, this constant is chosen as being equal to $(-1/g)$, and the coordinate s is then noted π and termed as "mass-based" coordinate (in the shallow atmosphere framework; for definition of mass-based coordinates in the deep atmosphere framework, not discussed in more detail here, see Wood and Staniforth, QJRMS, 2003).
We then have:

$$
\frac{\partial \pi}{\partial z}=-\rho_\bullet g \tag{16}
$$

Note that the definition of the coordinate π depends on the choice of the density $\rho_\bullet$ and should therefore be noted $\pi_\bullet$ rigorously speaking. However, the subscript $\bullet$ is omited for conciseness. The pseudo-vertical velocity in π coordinate is noted $\omega_\bullet$:

$$
\omega_\bullet := \frac{d\pi}{dt_\bullet} \tag{17}
$$

The continuity equation then becomes:

$$
\left.\frac{\partial u_\bullet}{\partial x}\right|_\pi+\left.\frac{\partial v_\bullet}{\partial y}\right|_\pi+\frac{\partial \omega_\bullet}{\partial \pi}=\left(\frac{\dot{R}_\bullet}{\rho_\bullet}\right) \tag{18}
$$

This continuity equation is "diagnostic" in the sense that in a numerical model, the source term $\dot{R}_\bullet$ would be diagnosed from other fields and/or from physical parameterisations.

In this paper, the upper boundary is assumed to be material and elastic at the zero pressure level. This reflects the choice of a physically unbounded domain in height. Without entering into more details, this condition is admitted to result in:

$$
\pi_T=p_T=0 \quad \text{and} \quad \omega_\bullet(\pi=0)=0 \tag{19}
$$

It should be noted that this is a loss of generality and that other choices for the upper boundary condtion (UBC) would be possible (non-zero pressure UBC, non-material UBC, non-elastic UBC, ,...) but they are not really in the scope of this paper. Further details about possible alternative boundary conditions can be found in the Scientific NH Documentation (they are however not implemented in Aladin or AROME models)

Using $\omega_\bullet(\pi=0)=0$, the pseudo vertical velocity $\omega_\bullet$ in π coordinate, obtained by integration of (18) from the top, writes:

$$
\omega_\bullet=-\int_0^\pi\left(\mathbf{\nabla}_\pi.\mathbf{V}_\bullet+\frac{\dot{R}_\bullet}{\rho_\bullet}\right)d\pi' \tag{20}
$$

where $\mathbf{\nabla}_\pi=(\partial/\partial x|_\pi\,,\partial/\partial y|_\pi)$, and $\mathbf{V}_\bullet=(u_\bullet,v_\bullet)$.

The transformation rules for the coordinate π are then:

$$\begin{aligned}\frac{\partial}{\partial z} &= -\rho_\bullet g \frac{\partial}{\partial \pi} &(21)\\ (\boldsymbol{\nabla})_z &= (\boldsymbol{\nabla})_\pi + \rho_\bullet g(\boldsymbol{\nabla}_\pi z)\frac{\partial}{\partial \pi} &(22)\end{aligned}$$

4 Transformation to hybrid terrain-following mass-based coordinate

A new hybrid coordinate η associated with the mass-based coordinate π defined in the previous section is defined by:

$$\pi(\eta) = A(\eta) + B(\eta)\pi_s \tag{23}$$

where A and B are "arbitrary" functions (they are not totally arbitrary in fact, depending on the minimum possible value of π_s). Note that rigorously, this new coordinate, depending of the choice of the density $\rho_\bullet$ should be noted $\eta_\bullet$. However similarly to the π coordinate, the subscript "$\bullet$" is omited throughout, unless explicitly specified. The domain boundaries are defined conventionally by $\eta = 0$ at top where $\pi = 0$, and $\eta = 1$ at the bottom where $\pi = \pi_s$, yielding:

$$\begin{aligned}A(0) = B(0) &= 0 &(24)\\ A(1) = B(1) - 1 &= 0 &(25)\end{aligned}$$

The upper boundary is assumed to be always material, which together with $\pi_{(\eta=0)} = 0$, leads to (see NH Documentation):

$$\dot{\eta}_{(\eta=0)} = 0 \tag{26}$$

Note that this notation $\dot{\eta}$ refers in fact to $(d\eta/dt_\bullet)$. The notation m is introduced:

$$m(\eta) := \frac{\partial \pi}{\partial \eta} \tag{27}$$

The transformation rules from π to η are:

$$\begin{aligned}\frac{\partial}{\partial \pi} &= \frac{1}{m}\frac{\partial}{\partial \eta} &(28)\\ (\boldsymbol{\nabla})_\pi &= (\boldsymbol{\nabla})_\eta - \frac{1}{m}(\boldsymbol{\nabla}_\eta \pi)\frac{\partial}{\partial \eta} &(29)\end{aligned}$$

The subscript "η" for the $\boldsymbol{\nabla}$ operator is dropped from now on, The Lagrangian derivative writes:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{V}_\bullet \boldsymbol{\nabla} + (m\dot{\eta})\frac{\partial}{\partial \eta} \tag{30}$$

We have:

$$\rho_\bullet \frac{\partial z}{\partial \eta} = -\frac{m}{g} \tag{31}$$

The continuity equation then writes:

$$\frac{\partial m}{\partial t} + \boldsymbol{\nabla} m \mathbf{V}_\bullet + \frac{\partial}{\partial \eta}(m\dot{\eta}) = \frac{m\dot{R}_\bullet}{\rho_\bullet} \tag{32}$$

By vertically integrating through the whole domain $\eta \in [0,1]$, we obtain an equation for the tendency of the hydrostatic surface pressure:

$$\frac{\partial \pi_s}{\partial t} = -\int_0^1 \boldsymbol{\nabla} m \mathbf{V}_\bullet d\eta + \int_0^1 \frac{m\dot{R}_\bullet}{\rho_\bullet} d\eta - [m\dot{\eta}]_s \tag{33}$$

where $[m\dot{\eta}]_s$ is the value of $m\dot{\eta}$ at the surface. This latter term is linked to mass-fluxes of the matter attached to $\rho_\bullet$ when the lower surface is not material for this matter. In effect, this term describes the evolution of the surface pressure due to surface processes in the absence of horizontal wind and of internal mass sources of $\rho_\bullet$. Therefore, if the lower surface is material for $\rho_\bullet$ this term will vanish, whereas if the surface is not material for $\rho_\bullet$ a source term will exist.

In a numerical model, the evaluation of the source term for π_s at the surface is based on mass-fluxes provided by the physical packages. The increase of pressure in a unit-area vertical column is equal to the increase of weight due to the mass transfer across the corresponding ground surface. If the lower surface is not material for a species k [with ρ_k, (u_k, v_k, w_k)], the mass flux at the surface writes $F_k(\eta = 1) = [-\rho_k w{*}_k]_{(\eta=1)}$ where $w{*}_k$ is the velocity of the species k along the direction orthogonal to the surface (for a species for which the surface is material, this velocity is zero by definition). In this case, the source term for π_s simply writes:

$$[m\dot{\eta}]_s = -g \sum_k F_k \tag{34}$$

where the sum involve all species for which the lower surface is not material.

Note: *The fact to express the source of the surface pressure in such a way implicitly means that this mass-flux of matter through the ground F_k is expressed in terms of mass transfer per unit of time per unit of <u>horizontal</u> ground surface area. However, it is not sure that all current the physics packages deliver such fluxes per unit surface of horizontally projected terrain. Similarly, it is not sure that $w{*}_k$ is not replaced by w_k for the computation of F_k in some physics packages. As a consequence, the hypothesis of a small slope is therefore hidden somewhere when using the fluxes provided by some physics packages directly as the input for the dynamics. The consequences of this discussion are not developped in more details in this paper.*

By vertically integrating through the domain $\eta' \in [0, \eta]$, we obtain:

$$m\dot{\eta} = \left[B \int_0^1 \boldsymbol{\nabla} m \mathbf{V}_\bullet d\eta' - \int_0^\eta \boldsymbol{\nabla} m \mathbf{V}_\bullet d\eta'\right] - \left[B \int_0^1 \frac{m\dot{R}_\bullet}{\rho_\bullet} d\eta' - \int_0^\eta \frac{m\dot{R}_\bullet}{\rho_\bullet} d\eta'\right] + B[m\dot{\eta}]_s \tag{35}$$

Finally the pseudo vertical velocity is obtained by applying the definition of the Lagrangian derivative (30) to the variable π:

$$\omega_\bullet = \mathbf{V}_\bullet \boldsymbol{\nabla} \pi - \int_0^\eta \boldsymbol{\nabla} m \mathbf{V}_\bullet d\eta' + \int_0^\eta \frac{m\dot{R}_\bullet}{\rho_\bullet} d\eta' \tag{36}$$

It is easily checked that at the surface, this writes:

$$\omega_\bullet(\eta = 1) = \frac{\partial \pi_s}{\partial t} + [\mathbf{V}_\bullet]_s \boldsymbol{\nabla} \pi_s + [m\dot{\eta}]_s \tag{37}$$

This equation expresses the fact that if the lower boundary is not a material surface for the matter $\rho_\bullet$ subjected to the velocity $(u_\bullet, v_\bullet, w_\bullet)$, the cross-surface pseudo-velocity component $\dot{\eta}$ must be taken into acount in the computation of $\omega_\bullet$ at the surface.

5 "Optimal" mass-based and hybrid coordinates

The equation (18) can actually be made purely diagnostic (even for the RHS) if a proper choice for the definition of $(\rho_\bullet, u_\bullet, v_\bullet, w_\bullet)$ is made for the considered atmosphere, in such a way that $\dot{R}_\bullet$ identically vanishes.

As shown in Malardel, in the case of a moist atmosphere with non-airborne (precipitating) condensated components, $\dot{R}_\bullet$ can be made identically vanishing by choosing the following definition:

$$\rho_\bullet := \rho \quad \text{and} \quad (u_\bullet, v_\bullet, w_\bullet) := (u, v, w) \tag{38}$$

where ρ is the total density of the atmosphere (including precipitating species), (u, v, w) is the so-called "barycentric" velocity corresponding to this total density (i.e. the mass averaged velocity of all components in ρ). The situation examined in section 12.3 with a "pseudo-adiabatic approximation for precipitations" also enters this case.

Remark: *The other possible definition ($\rho_\bullet = \rho_a$, $\mathbf{v}_\bullet = \mathbf{v}_a$), mentionned in section 2, which also makes $\dot{R}_\bullet$ identically vanishing for a precipitating atmopsphere, is no longer described here, since this path is not priviledged in Malardel's paper (essentially because it makes more difficult the formulation of the "dynamical equilibrium" assumption for the derivation of the momentum equation).*

In the case where $\dot{R}_\bullet$ identically vanishes, the continuity equation becomes:

$$\left.\frac{\partial u_\bullet}{\partial x}\right|_\pi + \left.\frac{\partial v_\bullet}{\partial y}\right|_\pi + \frac{\partial \omega_\bullet}{\partial \pi} = 0 \tag{39}$$

The form of this continuity equation is exactly similar to the usual form for dry or non-precipitating atmospheres. All vertically integrated versions of this local continuity equation keep their "traditional" form as for mono-component dry-air meteorology, thus leading to traditional expressions for $\omega_\bullet$, $(\partial \pi_s / \partial t)$ as in Laprise, 1992. Namely the above equation (20) reduces to:

$$\omega_\bullet = -\int_0^\pi \boldsymbol{\nabla}_\pi . \mathbf{V}_\bullet d\pi' \tag{40}$$

Similarly to the previous section a new hybrid coordinate η associated with this mass-based coordinate π is introduced. We have:

$$\rho \frac{\partial z}{\partial \eta} = -\frac{m}{g} \tag{41}$$

The continuity equation then writes:

$$\frac{\partial m}{\partial t} + \boldsymbol{\nabla} m \mathbf{V}_\bullet + \frac{\partial}{\partial \eta}(m\dot{\eta}) = 0 \tag{42}$$

By vertically integrating through the whole domain $\eta \in [0, 1]$, we obtain an equation for the tendency of the hydrostatic surface pressure:

$$\frac{\partial \pi_s}{\partial t} = -\int_0^1 \boldsymbol{\nabla} m \mathbf{V}_\bullet d\eta - [m\dot{\eta}]_s \tag{43}$$

where $[m\dot{\eta}]_s$ is the value of $m\dot{\eta}$ at the surface.
By vertically integrating through the domain $\eta' \in [0, \eta]$, we obtain:

$$m\dot{\eta} = \left[B \int_0^1 \boldsymbol{\nabla} m \mathbf{V}_\bullet d\eta' - \int_0^\eta \boldsymbol{\nabla} m \mathbf{V}_\bullet d\eta' \right] + B[m\dot{\eta}]_s \tag{44}$$

Finally the pseudo vertical velocity is obtained by:

$$\omega_\bullet = \mathbf{V}_\bullet \boldsymbol{\nabla} \pi - \int_0^\eta \boldsymbol{\nabla} m \mathbf{V}_\bullet d\eta' \tag{45}$$

It is easily checked that at the surface, this writes:

$$\omega_\bullet(\eta = 1) = \frac{\partial \pi_s}{\partial t} + [\mathbf{V}_\bullet]_s \boldsymbol{\nabla} \pi_s + [m\dot{\eta}]_s \tag{46}$$

6 Organized diffusive fluxes

In the case of a multi-component continuous fluid, the velocities of all components are not necessarily identical. If all components are gaseous or even contains only microscopic (airborne) condensates, all velocities can be considered as equal to a very good degree of approximation, but for a multi-phasic fluid as the atmospheric one, the presence of e.g. precipitating liquid water (treated as a continuous fluid) makes it relevant to consider different velocities for gases (or airborne components) and precipitations, at least at small scales or high resolutions. In continuous multi-components fluid mechanics, the motion of one component "k" relative to another component "l" is traditionally called "diffusion of k in l".

It is to be strongly outlined that this acceptation of the word "diffusion" is not similar to its traditional meaning in meteorology (where implicitly, the word "diffusion" rather contains the idea of a redistribution tending to attenuate gradients, due to molecular dissipation or turbulence processes).

The source terms linked to this multi-component diffusion may be identified in the local budget equation for a barycentric general property ψ in Malardel's, as e.g; the second term in RHS of her Eq. (1.15). They may be viewed as the sum of the divergence of multi-component diffusion fluxes given by: $\rho_k \psi_k \tilde{\mathbf{u}}_k$ in her notations.

Due to the historical inconsistency of the meaning of the word "diffusion" in meteorology and in multi-component fluid mechanics, we suggest, for multi-components meteorology, to use the term "organised diffusive fluxes" for these source terms, in order to avoid confusion.

Due to the specificities and scales of multi-components meteorology, which are discussed in more details in Malardel's paper, the organised diffusive terms are currently restricted to a vertical divergence of vertical fluxes only, and therefore simplify to:

$$-\frac{\partial \rho_k \psi_k \tilde{w}_k}{\partial z} \tag{47}$$

The transformation of these fluxes to mass-coordinate is straightforward:

$$+\frac{\rho g}{m}\frac{\partial \rho_k \psi_k \tilde{w}_k}{\partial \eta} \tag{48}$$

Since they always have this form, these fluxes are no longer detailed in the following. The exact status of these organized diffusive terms (physical or dynamical terms) may be questioned: by their very simple mathematical shape and the fact that they deal only with resolved fields, they may be viewed as dynamical, but in fact they are not dynamical in the sense that their computation requires the knowledge of the parameterised relative velocity $\tilde{w}_k$ (from the parameterised precipitation fallout velocity).

7 State equation

The state equation writes:

$$p = \rho_{av} R_{av} T \tag{49}$$

where $\rho_{av} = \rho_a + \rho_v$ is the density of the gaseous mixture, and R_{av} is the gas constant for the mixture, defined by:

$$R_{av} = \frac{\rho_a R_a + \rho_v R_v}{\rho_{av}} \tag{50}$$

In Aladin/AROME, the above state equation is rewritten as:

$$p = \rho R T \tag{51}$$

where the perfect gas constant R is given by:

$$R = \frac{\rho_{av}}{\rho} R_{av} \tag{52}$$

The total density ρ is context-dependent in the model: if the condensated species are prognostic variables, their density is taken into account in ρ.

8 Frameworks chosen for writing the equation systems

As outlined in the introduction, the definition of the mass-based coordinate itself depends on the choice made for the density appearing in the Laplace equation, linking the hydrostatic pressure (i.e. the "mass" field under the thin-layer approximation) to the height. In a numerical model, this density is in turn linked to the type of physics actually used in the parameterisation package.
In the Aladin/AROME/ARPEGE galaxy, the chronological order of relevant definitions can be summarized as followss:

- Define the value of a "total" R gaz-constant, as a function of the context (including or not such and such species, depending on the type of physics atually used). The resulting R is defined in such a way that in all cases: $\rho R = \rho_{av} R_{av}$, where ρ is the total density of species which have a physical existence in the model.
- Actually define the density ρ from the state equation (i.e. $\rho = p/RT$).
- This density ρ is then used everywhere in the dynamical part of the model, including in the definition of the coordinate.

We see that fundamentally, the choice of the coordinate in Aladin/AROME, is conditioned by the choice made for the definition of R (and thus of ρ), which in turn is conditioned by the species which have a physical existence in the model (i.e. which are treated prognostically).
Various choices can be made for the physical framework used to describe the atmosphere. In Aladin, these choices involve hydrostatic equations vs. fully elastic compressible equations, instantaneous fallout vs. suspension of precipitations, etc. In the following, we will restrict to some of the most used combinations of frameworks allowed in Aladin.

9 Case of a non-hydrostatic atmosphere with precipitations in suspension

This is the most general case examined in the paper by Malardel. The hydrostatic approximation is not retained and the precipitations are not assumed to be subjected to an instantaneous fallout. As a first consequence, the precipitating species must be described through a prognostic variable. As a second consequence, there are dynamical interactions between precipitating and non-precipitating components.
This case is developed in details in Malardel's paper, for the choice of the total density ρ and the total barycentric velocity (u, v, w) associated with this matter. We start here from the final equations obtained therein, in order to transform the equation set into the mass coordinate. This case correspond to the AROME model, or to a combination of the Aladin-NH model with any physics allowing suspended precipitations (as e.g. the "Lopez" scheme).

Remark: It should be noted that for such a degree of sophistication, the approximation of a constant atmospheric mass (so called $\delta m = 0$ option) is not very relevant, and will therefore not be discussed for this case.

9.1 Continuity equations

Since the total density and the total barycentric velocity are chosen for describing the dynamics of the parcels, the source terms $\dot{R}$ identically vanish in the the local continuity equation. Therefore, the various versions of the continuity equation (43)–(45) developed in section 5 remain valid:

$$\frac{\partial \pi_s}{\partial t} = -\int_0^1 \boldsymbol{\nabla} m\mathbf{V} d\eta - [m\dot{\eta}]_s$$

$$m\dot{\eta} = \left[B\int_0^1 \boldsymbol{\nabla} m\mathbf{V} d\eta' - \int_0^\eta \boldsymbol{\nabla} m\mathbf{V} d\eta'\right] + B[m\dot{\eta}]_s$$

$$\omega = \mathbf{V}\boldsymbol{\nabla}\pi - \int_0^\eta \boldsymbol{\nabla} m\mathbf{V} d\eta'$$

It is seen that in these latter equations, the only physical source is $[m\dot{\eta}]_s$ due to the non-materiality of the surface for some species.
In the case of the AROME model, $[m\dot{\eta}]_s$ includes the contributions of mass fluxes for liquid and solid precipitations $(\mathsf{R}+\mathsf{S})_{(\eta=1)}$ and of the mass flux of water-vapour by evaporation or dew deposition $\mathsf{E}_{(\eta=1)}$. The mass flux of dry gases is assumed to be zero, as well as for atmospheric dry aerosols. The mass flux linked to the deposition of cloud (fog) water to the ground is also assumed to be zero in the current AROME physics (however there exist a project for such a parameterisation in Meso-NH physics in a mid-term future). Finally, for the AROME model, we have:

$$[m\dot{\eta}]_s = g(\mathsf{R}+\mathsf{S}+\mathsf{E})_{(\eta=1)} \tag{53}$$

This leads to equations consistent with those in Catry's paper.

9.2 Equation for a given species

The local equation for the density of a given species k is directly derived from Malardel's (1.22):

$$\frac{d\rho_k}{dt} = -\rho_k D_3 - \frac{\partial \rho_k \tilde{w}_k}{\partial z} + (\dot{\rho_k})_{\text{phase}} \tag{54}$$

where the second RHS term is an organized diffusive flux, and the last RHS term is the source of ρ_k due to all phase changes involving the species k.

9.3 Horizontal momentum equation

Starting from (1.35) in Malardel, we have:

$$\rho\frac{d\mathbf{V}}{dt} = -\rho f\mathbf{k}\times\mathbf{V} - \boldsymbol{\nabla}_z p + [\text{viscous tensor term}] \tag{55}$$

Using the transformation rules, we obtain:

$$\frac{d\mathbf{V}}{dt} = -\frac{1}{\rho}\boldsymbol{\nabla}_\pi p - \frac{\partial p}{\partial \pi}\boldsymbol{\nabla}_\pi \phi - f\mathbf{k}\times\mathbf{V} + [\text{viscous tensor term}] \tag{56}$$

9.4 Vertical momentum equation

The starting local equation (1.40) in Malardel may be rewritten as:

$$\frac{dw}{dt} = -g - \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{1}{\rho}\text{div}(\sigma_w) - \frac{1}{\rho}\frac{\partial\left(\sum_k \rho_k \tilde{w}_k^2\right)}{\partial z} \tag{57}$$

Using the transformation rules, we obtain:

$$\frac{dw}{dt} = g\left(\frac{\partial p}{\partial \pi} - 1\right) + \frac{1}{\rho}\text{div}(\sigma_w) + g\frac{\partial\left(\sum_k \rho_k \tilde{w}_k^2\right)}{\partial \pi} \tag{58}$$

9.5 Thermodynamics equation

The constant pressure specific heat c_p has been defined in Malardel :

$$c_p := \frac{1}{\rho} \sum_k \rho_k c_{pk} \tag{59}$$

Additionally, we define:

$$\kappa := \frac{\rho_{av} R_{av}}{\rho c_p} \tag{60}$$

The thermodynamics equation drawn from (1.59) of Malardel then writes:

$$\frac{dT}{dt} = \frac{1}{\rho c_p} \left[\frac{dp}{dt} - \sum_k \rho_k \tilde{w}_k \frac{\partial c_{pk} T}{\partial z} + \epsilon + L_v(T)\dot{\rho}_l + L_i(T)\dot{\rho}_i \right] \tag{61}$$

which writes finally:

$$\frac{dT}{dt} = \frac{1}{\rho c_p} \left[\frac{dp}{dt} + \rho Q_1 \right] \tag{62}$$

where:

$$Q_1 := \frac{1}{\rho} \left[- \sum_k \rho_k \tilde{w}_k \frac{\partial c_{pk} T}{\partial z} + \epsilon + L_v(T)\dot{\rho}_l + L_i(T)\dot{\rho}_i \right] \tag{63}$$

Using the state equation, (62) can be rewritten as:

$$\frac{dT}{dt} = \frac{\kappa T}{p} \frac{dp}{dt} + \frac{Q_1}{c_p} \tag{64}$$

From the state equation, we have:

$$\frac{1}{p}\frac{dp}{dt} = \frac{1}{\rho_{av}} \frac{d\rho_{av}}{dt} + \frac{1}{R_{av}} \frac{dR_{av}}{dt} + \frac{1}{T}\frac{dT}{dt} \tag{65}$$

Besides, from (54) we have:

$$\frac{d\rho_{av}}{dt} = -\rho_{av} D_3 - \frac{\partial \rho_{av} \tilde{w}_{av}}{\partial z} + (\dot{\rho_{av}})_{\text{phase}} \tag{66}$$

where $(\dot{\rho_{av}})_{\text{phase}}$ is the source of ρ_{av} due to phase changes. This equation is a particular form of (54) applied to $k = av$. Combining these two latter equations, we obtain:

$$\frac{1}{p}\frac{dp}{dt} = -D_3 - \frac{1}{\rho_{av}} \frac{\partial \rho_{av} \tilde{w}_{av}}{\partial z} + \frac{1}{\rho_{av}} (\dot{\rho_{av}})_{\text{phase}} + \frac{1}{R_{av}} \frac{dR_{av}}{dt} + \frac{1}{T}\frac{dT}{dt} \tag{67}$$

Which rewrites:

$$\frac{1}{p}\frac{dp}{dt} = -D_3 + \frac{Q_2}{\kappa c_p T} + \frac{1}{T}\frac{dT}{dt} \tag{68}$$

where:

$$Q_2 := \kappa c_p T \left[-\frac{1}{\rho_{av}} \frac{\partial \rho_{av} \tilde{w}_{av}}{\partial z} + \frac{1}{\rho_{av}} (\dot{\rho_{av}})_{\text{phase}} + \frac{1}{R_{av}} \frac{dR_{av}}{dt} \right] \tag{69}$$

Now, inserting (68) in (64), we obtain:

$$\frac{dT}{dt} = -\frac{\kappa}{1-\kappa} T D_3 + \frac{Q}{(1-\kappa) c_p} \tag{70}$$

where:

$$Q := Q_1 + Q_2. \tag{71}$$

The dynamical part of this equation keeps a form similar to the traditional form valid for a dry atmosphere [cf. e.g. Laprise's eq. (23')]. It should be noted that $\kappa/(1-\kappa)$ is a "moist equivalent" of the traditional coefficient (R/c_v), and $(1-\kappa)c_p$ is a "moist equivalent" of c_v. Equation (70) expresses the form of the thermodynamics equation in mass coordinate for a precipitating atmosphere. The operator (d/dt) and the quantity D_3 must therefore be understood here as expressed in mass coordinate. The diabatic term Q also needs to be expressed in mass coordinate, which is not pursued here, because this is a purely "technical" task.

10 Case of an hydrostatic atmosphere with precipitations in suspension

In this framework, compared to the previous most general framework, we simply retain the hydrostatic approximation. However, unlike in the current operational (hydrostatic) version of Aladin, the condensated species are taken into account in the dynamical part of the model (through their inertia, their drag etc.). This framework reflects the case of the hydrostatic version of Aladin when used with prognostic variables for condensated species allowed to remain in suspension (e.g. Aladin-H used with AROME physics, or with the "Lopez" scheme).
The local vertical momentum equation (58) is rewritten as:

$$\frac{dw}{dt} - \frac{1}{\rho}\mathrm{div}(\sigma_w) - g\frac{\partial\left(\sum_k \rho_k \tilde{w}_k^2\right)}{\partial\pi} = g\frac{\partial p}{\partial\pi} - g \tag{72}$$

The introduction of the hydrostatic approximation is performed by stipulating that all LHS terms are negligible with respect to any of the (dominant) individual term of the RHS. As a consequence, this equation is replaced by a diagnostic equation :

$$\frac{\partial p}{\partial\pi} = 1 \tag{73}$$

Without loss of generality for a model with an elastic upper boundary condition (cf NH Documentation), this equation may in turn be transformed into:

$$p = \pi \tag{74}$$

Therefore, we have:

$$p = \pi = \rho_{av} R_{av} T \tag{75}$$

and:

$$\frac{\partial p}{\partial z} = \frac{\partial\pi}{\partial z} = -\rho g \tag{76}$$

It should be noted that is spite of appearances, these two latter equations are not redundant or incompatible. Their simultaneous validity simply means that the pressure profile, the thermal profile and the ρ_{av} and ρ profiles are linked together in the case of an hydrostatic atmosphere. For an atmosphere allowing condensates, the additional degree of freedom represented by $\rho \neq \rho_{av}$ implies an additional equation compared to the more classical case of an hydrostatic atmosphere without precipitations allowed in the parcels. In the case of an hydrostatic atmosphere allowing condensates, the Laplace's equation thus becomes:

$$\frac{\partial \ln\pi}{\partial z} = -\frac{\rho}{\rho_{av}}\frac{g}{R_{av}T} \tag{77}$$

Using the same barycentric wind and total density as in the previous section, the local continuity equation is unchanged. The various versions of the vertically integrated continuity equation (43)–(45) developed in section 5 remain valid:

$$\frac{\partial \pi_s}{\partial t} = -\int_0^1 \boldsymbol{\nabla} m\mathbf{V} d\eta - [m\dot{\eta}]_s$$

$$m\dot{\eta} = \left[B \int_0^1 \boldsymbol{\nabla} m\mathbf{V} d\eta' - \int_0^{\eta} \boldsymbol{\nabla} m\mathbf{V} d\eta' \right] + B[m\dot{\eta}]_s$$

$$\omega = \mathbf{V}\boldsymbol{\nabla}\pi - \int_0^{\eta} \boldsymbol{\nabla} m\mathbf{V} d\eta'$$

As in the previous section, since we include all species in the definition of the density, we have:

$$[m\dot{\eta}]_s = g(\mathsf{R} + \mathsf{S} + \mathsf{E})_{(\eta=1)} \tag{78}$$

The thermodynamics equation (64) can be transformed to the hydrostatic framework by using (40):

$$\frac{dT}{dt} = \frac{\kappa T \omega}{p} + \frac{Q_1}{c_p} \tag{79}$$

Note that the diabatic term here is not Q but actually Q_1.

The subsequent consequences of the hydrostatic approximation in this framework are purely "technical" and are no futher developed here. They mainly consist in formally replacing p by π at any occurence in the equation system.

11 The pseudo-adiabatic approximation for precipitations

The term "pseudo-adiabatic approximation for precipitations" is introduced by Malardel to describe the framework used in the currently operational (hydrostatic) version of ARPEGE or Aladin. The main characteristics of this approximation is twofold:

- Any condensate is assumed to be precipitating. Equivalently, this assumption states that there is no cloud condensate in the system.
- Precipitating species are assumed to fall instantaneously at ground along a vertical fall path, with possibly some mass-exchanges (re-evaporation) with the vapour species in case of unsaturated layers in the fall path below the location of generation.

It is therefore seen that the "pseudo-adiabatic approximation for precipitations" is the combination of "no cloud" and "instantaneous fallout" assumptions.
The "no cloud" assumption writes:

$$\rho_c \equiv 0 \quad \text{i.e.} \quad \rho \equiv \rho_{av} + \rho_p \tag{80}$$

where ρ_p is the density of precipitating species.

Note: *the case of a system in which only the "instantaneous fallout" assumption is made (and therefore in which cloud water density is described) can be easily derived from the case examined in this section by a simple formal replacement of ρ_{av} into $\rho_{avc} = \rho_{av} + \rho_c$ in all subsequent formulae of this section.*

The assumption of instantaneous fallout is now expressed mathematically. The general continuity equation for precipitations writes:

$$\frac{\partial \rho_p}{\partial t} = -\boldsymbol{\nabla}.(\rho_p \mathbf{v}_p) + [\dot{\rho}_p]_{\text{phase}} \tag{81}$$

According to the instantaneous fallout assumption, the fall path is assumed to be vertical, i.e. $(u_p = v_p \equiv 0)$, and the precipitation is instantaneously evacuated to the ground, hence the density of precipitation (and its local tendency) remains almost zero at every time in any layer, that is:

$$\frac{\partial \rho_p}{\partial t} \approx 0 \quad \text{and} \quad \rho_p \approx 0 \tag{82}$$

Using these assumptions leads to:

$$[\dot{\rho}_p]_{\text{phase}} \approx \frac{\partial (\rho_p w_p)}{\partial z} \tag{83}$$

The vertical precipitation flux $(-\rho_p w_p)$ is noted P, and we may finally write:

$$[\dot{\rho}_p]_{\text{phase}} \approx -\frac{\partial \mathsf{P}}{\partial z} \tag{84}$$

The instantaneous fallout approximation allows to write the phase-change tendency in terms of the divergence of the vertical flux of precipitations. The approximation may therefore be viewed as a instantaneous balance between the net generation of precipitations and the vertical divergence of the precipitation flux, in order to always maintain a vanishing density of precipitations.
Returning to the second equality in (82), and to (83), we see that for finite generation of precipitation $[\dot{\rho}_p]_{\text{phase}}$ to take place, the vertical velocity of precipitations must be unbounded (which by the way is consistent with the instantaneous fallout assumption). The velocity w_p is therefore ill-defined and should not be used, but the vertical flux of precipitation P is well-defined.

The phase-change tendency for the vapour is opposite the latter due to the "no cloud" assumption:

$$[\dot{\rho}_{av}]_{\text{phase}} \approx \frac{\partial \mathsf{P}}{\partial z} \tag{85}$$

Remark: *It should be noted that even if the local tendency of ρ_p and ρ_p itself always remain zero, the source of ρ_p due to phase-changes is non-zero. It is balanced at any time by the vertical flux of precipitations*

We discuss now the domain in which the pseudo-adiabatic approximation is likely to be valid for a given application. As seen in the beginning of the discussion, the precipitations are assumed to be evacuated instantaneously. For a numerical model this implicitly means that the fallout time is smaller than the time-step of the model. The approximation is therefore likely to be valid for models with let say $\Delta t \approx$ 0.5–1 h. For models with increasingly smaller time-steps, the validity of the assumption becomes increasingly questionable, especially if the horizontal resolution becomes so accurate that large upward velocities are likely encountered. Another conceptual problem with the approximation is that it assumes an arbitrarily large fall-velocity for the precipitations. Since the density is zero, the vertical momentum of precipitations is bounded and quite well-defined by construction. However, the kinetic energy of the precipitations is virtually unbounded due to their assumed unbounded fall-velocity. When deriving an energetic invariant in the context of this assumption, we are therefore obliged to ignore the contribution of the precipitations. This neglection of vertical kinetic aspects seems to be more adpated to an hydrostatic context, where all vertical kinetic aspects are also neglected. In other words, taking into account the subtleties linked to the vertical kinetic energy of the gas in the nonhydrostatic framework while neglecting those aspects for the precipitations may seem strange, especially when considering that even if they have smaller densities, the precipitations may have large fall velocities compared to the vertical velocity of the gases. As a consequence, the pseudo-adiabatic approximation seems to be better adapted to the hydrostatic framework. Anyway this is not inconsistent with the above discussion about time- and space-scale. We should not encourage to use the approximation when nonhydrostatic effects are expected to be significant. For this reason, the case of a nonhydrostatic model with the pseudo-adiabatic approximation for precipitations is not discussed furthermore in this note.

12 Case of a hydrostatic atmosphere with "pseudo-adiabatic" approximation for precipitations

This case is the one which contains the most severe set of assumptions among the variety of frameworks examined in this paper: both the hydrostatic approximation and the "pseudo-adiabatic" approximation for precipitations are retained.
As discussed above, a mandatory consequence of the "pseudo-adiabatic approximation for precipitations" is that the continuity equation only involves the gaseous density ρ_{av}. The most natural framework for writing the equations is then to choose $\rho_{\bullet} = \rho_{av}$ and $\mathbf{v}_{\bullet} = \mathbf{v}_{av}$. This choice is the one which has been made in the old versions of Aladin, and is described in the sub-sections $\delta m = 0$ (for the constant total mass option) and "old" $\delta m = 1$ (for the variable total mass option) below.
However, since the density of precipitations is zero, there is an indeterminacy about the matter which is actually attached to this density: it may be considered as the gaseous species "av" or as the total fluid including precipitations "av+p". The possibility to consider ρ_{av} as the density of the total fluid is exploited in the so-called "new" $\delta m = 1$ option. In this case, the equations are written in the total barycentric framework (but still involve ρ_{av}) and the total barycentric velocity $\mathbf{v}_{\bullet} = \mathbf{v}$ is chosen. This choice is made in the "new" $\delta m = 1$ option, which is discussed at the end of this section.

Note: *The "old" $\delta m = 1$ option is to be removed from Aladin code in the next future, and replaced by the "new" $\delta m = 1$ one. Therefore the sub-section about "old" $\delta m = 1$ below, should be considered as almost obsolete*

12.1 "Old" option $\delta m = 1$

The case of the "old" $\delta m = 1$ option consists in choosing the gaseous velocity (u_{av}, v_{av}, w_{av}) to describe the dynamics. hence we have :

$$\rho_{\bullet} := \rho_{av} \quad \text{and} \quad (u_{\bullet}, v_{\bullet}, w_{\bullet}) := (u_{av}, v_{av}, w_{av}) \tag{86}$$

This case is described for height coordinates in Malardel's paper. Following Eq (2.8) therein, the local continuity equation writes:

$$\frac{d\rho_{av}}{dt_{av}} = -\rho_{av} D_{3\ av} + \frac{\partial \mathsf{P}}{\partial z} \tag{87}$$

where $D_{3\ av}$ is the 3D divergence of the gaseous wind flow, and P is the diagnostic vertical precipitation flux provided by the physical package. The possible evaporation of rain in unsaturated layers in the fall path is taken into account via a decrease of the magnitude of P in such layers.
We see that in this case the mass-coordinate associated with this continuity equation is not an "optimal" mass-coordinate, because the source term noted $\dot{R}_{\bullet}$ in (11) does not vanish here. However, a mass-coordinate can still be built and we have:

$$\dot{R}_{\bullet} = \frac{\partial \mathsf{P}}{\partial z} \tag{88}$$

The coordinate π is defined by:

$$\frac{\partial \pi}{\partial z} = -\rho_{av} g \tag{89}$$

The difference between (76) and (89) shows that the creation of precipitations in a layer (without re-evaporation below) instantaneously decreases the pressure in the whole column below, because the density of the gaseous component in this layer is decreased by the amount of the created precipitation. This instantaneous non-local decrease of hydrostatic-pressure is fictitious from the physical realism point of view. However, it is physically consistent with the assumption that the precipitation is instantaneously removed from the atmosphere by the

fallout process, which implies that the corresponding mass must not participate to the hydrostatic-pressure of the atmosphere.
In π coordinate, the continuity equation is:

$$\left.\frac{\partial u_{av}}{\partial x}\right|_{\pi}+\left.\frac{\partial v_{av}}{\partial y}\right|_{\pi}+\frac{\partial \omega_{av}}{\partial \pi}=-g\frac{\partial \mathsf{P}}{\partial \pi} \tag{90}$$

The pseudo vertical velocity ω_{av} coordinate writes:

$$\omega_{av}=-\int_0^{\pi}(\boldsymbol{\nabla}_{\pi}.\mathbf{V}_{av})d\pi'-g\mathsf{P}(\pi) \tag{91}$$

Since the precipitation fluxes are provided on layer interfaces in the Aladin vertical discretisation, they must be interpolated at the layer levels.

The transformation rules for the coordinate π are :

$$\frac{\partial}{\partial z} = -\rho_{av}g\frac{\partial}{\partial \pi} \tag{92}$$

$$(\boldsymbol{\nabla})_z = (\boldsymbol{\nabla})_{\pi}+\rho_{av}g(\boldsymbol{\nabla}_{\pi}z)\frac{\partial}{\partial \pi} \tag{93}$$

The local continuity equation in η coordinate writes:

$$\frac{\partial m}{\partial t}+\boldsymbol{\nabla}m\mathbf{V}_{av}+\frac{\partial}{\partial \eta}(m\dot{\eta})=-g\frac{\partial \mathsf{P}}{\partial \eta}(\eta) \tag{94}$$

The wind is the one associated with the considered matter, therefore the only sources for $[m\dot{\eta}]_s$ are those linked to the non-materiality of the surface for the considered matter ρ_{av}. For dry air, the surface is material, but for the water-vapour, a mass-flux at the surface $\mathsf{E}_{(\eta=1)}$ due to surface evaporation must be taken into account if it exist. Hence we have:

$$[m\dot{\eta}]_s=g\mathsf{E}_{(\eta=1)} \tag{95}$$

and the integrated continuity equations become:

$$\frac{\partial \pi_s}{\partial t}=-\int_0^1\boldsymbol{\nabla}m\mathbf{V}_{av}d\eta-g\mathsf{P}_{(\eta=1)}-g\mathsf{E}_{(\eta=1)}$$

$$m\dot{\eta}=\left[B\int_0^1\boldsymbol{\nabla}m\mathbf{V}_{av}d\eta'-\int_0^{\eta}\boldsymbol{\nabla}m\mathbf{V}_{av}d\eta'\right]+gB\mathsf{P}_{(\eta=1)}+gB\mathsf{E}_{(\eta=1)}-g\mathsf{P}(\eta)$$

$$\omega_{av}=\mathbf{V}\boldsymbol{\nabla}\pi-\int_0^{\eta}\boldsymbol{\nabla}m\mathbf{V}_{av}d\eta'-g\mathsf{P}(\eta)$$

In a fully barycentric context, the mutual interactions (dynamical drag, thermal exchanges) between precipitating and non-precipitating species would appear as the sum of the organized diffusion terms for all species. If the corresponding properties (momentum, enthalpy) are locally identical for the gaseous and precipitating species, then this sum vanishes due to the choice of the barycentric framework.
Here, the framework is not barycentric, therefore the organized diffusion terms do not vanish, even if the properties are equal for all components. These interactions are parameterized when the option $\delta m=1$ is used: the precipitation is assumed to instantaneously bring its properties (momentum, enthalpy) from the above origin layers to the current desposition layer during its fall. As a consequence the net interaction between precipitating and non-precipitating species in this non-barycentric framework is represented by the organised diffusion term of precipitations in gaseous component. The principle of the parameterization is to stipulate that:

$$\rho_p \tilde{w}_p = -\mathsf{P} \tag{96}$$

Note that this is consistent with the fact that the gaseous vertical velocities are assumed to be negligible with respect to the fallout velocity of the precipitations. The organised diffusion term for a property ψ_p linked to the precipitations then becomes:

$$-\frac{1}{\rho_{av}}\mathsf{P}\frac{\partial \psi_p}{\partial z} = -g\mathsf{P}\frac{\partial \psi_p}{\partial \pi} \tag{97}$$

The thermodynamics equation becomes:

$$\frac{dT}{dt_{av}} = \frac{\kappa T \omega_{av}}{\pi} - g\mathsf{P}\frac{\partial T_p}{\partial \pi} + \frac{Q_1'}{c_p} \tag{98}$$

in which Q_1' is given by:

$$Q_1' := \frac{1}{\rho}\left[\epsilon + L_v(T)\dot{\rho}_l + L_i(T)\dot{\rho}_i\right], \tag{99}$$

and T_p is the temperature of the precipitations.The parameterized mutual interaction is the second RHS term of (98). The same discussion applies to the momentum equation.

12.2 option $\delta m = 0$

This case is a further approximation of the previous one, in which any source term in the previous continuity equation is simply neglected. The density and velocity variables are still chosen as follows:

$$\rho_\bullet := \rho_{av} \quad \text{and} \quad (u_\bullet, v_\bullet, w_\bullet) := (u_{av}, v_{av}, w_{av}) \tag{100}$$

In height coordinate, the local continuity equation therefore writes, in the case $\delta m = 0$:

$$\frac{d\rho_{av}}{dt_{av}} = -\rho_{av} D_{3\ av} \tag{101}$$

Since $\dot{R}_\bullet$ vanishes in the above equation, the mass-coordinate obtained is an optimal one.
Conceptually, this case $\delta m = 0$ means that the density of the gaseous component is only governed by the divergence of the gaseous flow. The net budget of the vertical precipitation flux has no impact on the local density of the gaseous component, which is not physically realistic. This , means that any creation of precipitation is instantaneously replaced by an equivalent mass of gas. Moreover, since q_v is left unmodified during this replacement, this means that the replacing gas is in fact dry air.
As a consequence, (76) is replaced by:

$$\frac{\partial \pi}{\partial z} = -\rho_{av} g \tag{102}$$

The difference between (76) and (102) shows that the creation of precipitations has no impact on the hydrostatic pressure below, because the density of the gaseous component in this layer is not modified by the creation of precipitations, due to the fictitious introduction of dry air in replacement of the instantaneoulsy removed precipitation.
Equations (74) and (75), remain valid. Equation (40) becomes:

$$\omega_{av} = -\int_0^\pi \boldsymbol{\nabla}_\pi.\mathbf{V}_{av} d\pi' \tag{103}$$

The various versions of the continuity equation write:

$$\frac{\partial m}{\partial t} + \boldsymbol{\nabla} m \mathbf{V}_{av} + \frac{\partial}{\partial \eta}(m\dot{\eta}) = 0 \tag{104}$$

Since all sources in the continuity equations are neglected:

$$[m\dot{\eta}]_s = 0 \tag{105}$$

and the integrated continuity equationsbecome:

$$\frac{\partial \pi_s}{\partial t} = -\int_0^1 \boldsymbol{\nabla} m \mathbf{V}_{av} d\eta$$

$$m\dot{\eta} = \left[B \int_0^1 \boldsymbol{\nabla} m \mathbf{V}_{av} d\eta' - \int_0^\eta \boldsymbol{\nabla} m \mathbf{V}_{av} d\eta' \right]$$

$$\omega_{av} = \mathbf{V}_{av} \boldsymbol{\nabla} \pi - \int_0^\eta \boldsymbol{\nabla} m \mathbf{V}_{av} d\eta'$$

The thermodynamics equation becomes:

$$\frac{dT}{dt_{av}} = \frac{\kappa T \omega_{av}}{\pi} + \frac{Q_1}{c_p} \tag{106}$$

12.3 "New" option $\delta m = 1$

As seen above, when the pseudo-adiabatic approximation is used, the density of precipitation vanishes, but the momentum of the precipitations does not vanish. The total density is ρ_{av}. The horizontal velocity of precipitations is also assumed to be zero, and the total mass-averaged velocity (total barycentric velocity) then writes:

$$u := u_{av} \tag{107}$$

$$v := v_{av} \tag{108}$$

$$w := w_{av} + (\rho_p w_p / \rho_{av}) = w_{av} - (\mathsf{P}/\rho_{av}) \tag{109}$$

It should be noted that the barycentric wind is the one attached to the total matter and not to the gaseous components only.
The definitions relevant for the specification of the coordinate in this case are then:

$$\rho_\bullet := \rho_{av} \quad \text{and} \quad (u_\bullet, v_\bullet, w_\bullet) := (u, v, w) \tag{110}$$

The coordinate π is defined by:

$$\frac{\partial \pi}{\partial z} = -\rho_{av} g \tag{111}$$

The local continuity equation for ρ_{av} writes:

$$\frac{\partial \rho_{av}}{\partial t} = -\mathrm{div}(\rho_{av} \mathbf{v}_{av}) + \frac{\partial \mathsf{P}}{\partial z} \tag{112}$$

where P is the diagnostic vertical precipitation flux due to the water phase changes, provided by the physical package. The continuity equation may be rewritten as:

$$\frac{\partial \rho_{av}}{\partial t} = -\mathrm{div}(\rho_{av} \mathbf{v}) \tag{113}$$

or equivalently, in Lagrangian form:

$$\frac{d\rho_{av}}{dt} = -\rho_{av} \mathrm{div}(\mathbf{v}) \tag{114}$$

We see that in this case the mass-coordinate associated with this continuity equation is an "optimal" mass-coordinate.

Remark: *We could equivalently have derived the equation for the total density ρ:*

$$\frac{\partial \rho}{\partial t} = \frac{\partial \rho_{av}}{\partial t} + \frac{\partial \rho_p}{\partial t} = -\mathrm{div}(\rho_{av}\mathbf{v}_{av}) + [\dot{\rho_{av}}]_{\text{phase}} + \frac{\partial \mathsf{P}}{\partial z} + [\dot{\rho_p}]_{\text{phase}} \tag{115}$$

in this case we would have cancelled the two physical sources and obtained:

$$\frac{d\rho}{dt} = -\rho_{av}\mathrm{div}(\mathbf{v}) \tag{116}$$

All this is consistent with the fact that ρ and ρ_{av} are numerically equal.

Similarly to the "old" $\delta m = 1$ option, the creation of precipitations in a layer without re-evaporation below instantaneously decreases the pressure in the whole column below, because the density of the gaseous component in this layer is decreased by the amount of the created precipitation.
In π coordinate, the continuity equation is:

$$\left.\frac{\partial u}{\partial x}\right|_{\pi} + \left.\frac{\partial v}{\partial y}\right|_{\pi} + \frac{\omega}{\partial \pi} \ = 0 \tag{117}$$

The pseudo vertical velocity in π coordinate, ω writes:

$$\omega = -\int_0^{\pi} (\boldsymbol{\nabla}_{\pi}.\mathbf{V})d\pi' \tag{118}$$

The transformation rule from z to η along the vertical yields:

$$\rho_{av}\frac{\partial z}{\partial \eta} = -\frac{m}{g} \tag{119}$$

The local continuity equation writes:

$$\frac{\partial m}{\partial t} + \boldsymbol{\nabla} m\mathbf{V} + \frac{\partial}{\partial \eta}(m\dot{\eta}) = 0 \tag{120}$$

For the computation of the term $[m\dot{\eta}]_s$, we must consider the species of the system for which the lower boundary is not material. This is the case for the water-vapour, for which an evaporation at the ground may take place, and also for the precipitation, for which the mass-flux at the ground may be non-zero. Therefore, the total source term at the surface then writes:

$$[m\dot{\eta}]_s = g\mathsf{E}_{(\eta=1)} + g\mathsf{P}_{(\eta=1)} \tag{121}$$

and the integrated continuity equations become:

$$\frac{\partial \pi_s}{\partial t} = -\int_0^1 \boldsymbol{\nabla} m\mathbf{V}d\eta - g\mathsf{E}_{(\eta=1)} - g\mathsf{P}_{(\eta=1)}$$

The other versions of the continuity equation write:

$$m\dot{\eta} = \left[B\int_0^1 \boldsymbol{\nabla} m\mathbf{V}d\eta' - \int_0^{\eta} \boldsymbol{\nabla} m\mathbf{V}d\eta'\right] + gB\mathsf{E}_{(\eta=1)} + gB\mathsf{P}_{(\eta=1)}$$

$$\omega = \mathbf{V}\boldsymbol{\nabla}\pi - \int_0^{\eta} \boldsymbol{\nabla} m\mathbf{V}d\eta'$$

For the thermodynamics equation, we start from the one obtained in the "old" $\delta m = 1$ option and simply apply the change of reference frame:

$$\frac{d}{dt} = \frac{d}{dt_{av}} + g\mathsf{P}\frac{\partial}{\partial \pi} \tag{122}$$

Additionally we have:

$$\omega_{av} = \omega - g\mathsf{P} \tag{123}$$

Consequently, the thermodynamics equation becomes:

$$\frac{dT}{dt} = \frac{\kappa T \omega}{\pi} - \frac{\kappa T g\mathsf{P}}{\pi} - g\mathsf{P}\frac{\partial T}{\partial \pi} + \frac{Q_1'}{c_p} + g\mathsf{P}\frac{\partial T}{\partial \pi} \tag{124}$$

$\Longrightarrow$

$$\frac{dT}{dt} = \frac{\kappa T \omega}{\pi} - \frac{\kappa T g\mathsf{P}}{\pi} + \frac{Q_1'}{c_p} \tag{125}$$

It is seen in (125) that an additional term appears in relation with the conversion term. This term is due to the fact that the condensate part of the parcel must not be involved in the expansive cooling.

It also is seen that the organised diffusion terms for the gaseous and precipitating species mutually compensate. This can be seen as the consequence of the fact that the diffusion fluxes are parameterized in such a way to exactly compensate:

$$\rho_{av}\tilde{w}_{av} = \rho_{av}(w_{av} - w) = -g\mathsf{P} = -\rho_p\tilde{w}_p \tag{126}$$

This is consistent with the fact that the framework is a barycentric framework in the "new" $\delta m = 1$ option. The same feature applies to the organised diffusion of the momentum.

However, this is not consitent with the current code of the model, in which an enthalpy equilibrium is assumed instead of a thermodynamic equlibrium. What is currently coded in `CPTEND` implies that $h_p = h_{av}$ since the same deposition is applied for h than for u and v. Therefore, the current code is not consistent with the above development, and should be corrected.

13 Conclusions

The change of vertical coordinate from height to hybrid mass-based terrain-following coordinate in the multiphasic context is very similar to its monophasic counterpart. The only point which need a caution focus is the definition of the density and the derivation of the continuity equation, which is the starting basis for the change of coordinate.
This paper should allow to have a clear idea of how to write the dynamical part of the multiphasic system in mass-ccordinate, and also of how to correctly interface a given physical package to the dynamical system, taking into account the set of approximations used for the dynamics.