AROME Training Course: Time Stepping of NH model ALADIN

Jozef Vivoda SHMI

The content of this session

- Time stepping:
  - the procedure how to proceed from one time instant to the next one. It shall be stable, consistent and therefore convergent. It consist from advection and adjustment.

- Adjustmet:
  - solve the problem how to control the signals included in the non-hydrostatic dynamical system. This problem is relevant also in the resting atmosphere.

- Advection:
  - solve the problem how to treat advection process (doesn't exists in the resting atmosphere)

The content of session - time stepping

We will speak about:

1. description of problem
2. SI schemes
3. iterative schemes
   (a) partial iterative scheme
   (b) full iterative scheme

The nonlinear problem

We solve the time evolution of the non-linear problem

$$\frac{dX}{dt} = \mathcal{M}(X)$$

$\mathcal{M}$ - non-linear model $\qquad$ X - vector of prognostic variables

The $\mathcal{M}$ is formulated with the two possible sets of prognostic variables

Prog. variables with d_3

$$\begin{aligned} \vec{v} &= (u, v) \\ q &= ln(\pi_s) \\ T &= \\ d_3 &= -\frac{gp}{mRT}\frac{\partial w}{\partial \eta} \\ \hat{q} &= ln\left(\frac{p}{\pi}\right) \end{aligned}$$

Prog. variables with d_4

$$\begin{aligned} \vec{v} &= (u, v) \\ q &= ln(\pi_s) \\ T &= \\ d_4 &= \mathcal{X} - \frac{gp}{mRT}\frac{\partial w}{\partial \eta} \\ \hat{q} &= ln\left(\frac{p}{\pi}\right) \end{aligned}$$

The nonlinear problem

We use the symbolic notation $\mathcal{X}$ for the quantity

$$\mathcal{X} = \frac{p}{mRT}\vec{\nabla}\phi\frac{\partial\vec{v}}{\partial\eta}.$$

The differences between the two formulations are in the prognostic variable for vertical velocity related equation for d_3 and d_4.

$$\frac{dd_3}{dt} = -\frac{g^2p}{mR_dT}\frac{\partial}{\partial\eta}\left(\frac{1}{m}\frac{\partial(p-\pi)}{\partial\eta}\right) + \frac{gp}{mR_dT}\vec{\nabla}w\frac{\partial\vec{v}}{\partial\eta} - d_3D_3 + d_3D$$

$$\frac{dd_4}{dt} = -\frac{g^2p}{mR_dT}\frac{\partial}{\partial\eta}\left(\frac{1}{m}\frac{\partial(p-\pi)}{\partial\eta}\right) + \frac{gp}{mR_dT}\vec{\nabla}w\frac{\partial\vec{v}}{\partial\eta} - d_3D_3 + d_3D + \frac{d\mathcal{X}}{dt}$$

and diagnostic relations for elastic term D_3

$$D_3 = \vec{\nabla}\vec{v} + \mathcal{X} + d_3 \qquad D_3 = \vec{\nabla}\vec{v} + d_4$$

and diagnostic term for vertical velocity

$$\frac{\partial w}{\partial\eta} = -\frac{mRT}{gp}d_3 \qquad \frac{\partial w}{\partial\eta} = -\frac{mRT}{gp}(d_4 - \mathcal{X})$$

Evolution of d_4 quantity

Terms which contains $\vec{\nabla}\mathcal{X}$ and term $\frac{\partial \mathcal{X}}{\partial t}$ are time discretized in a different manner than the other model terms in order to save CPU time needed to compute second order spatial derivatives in spectral model.
Local evolution of d_4 is given by the equation

$$\frac{\partial d_4}{\partial t} = -\vec{v}\vec{\nabla}(d_4 - \mathcal{X}) - \dot{\eta}\frac{\partial (d_4 - \mathcal{X})}{\partial \eta} - \frac{g^2 p}{mRT}\frac{\partial}{\partial \eta}\left(\frac{1}{m}\frac{\partial (p-\pi)}{\partial \eta}\right) + \frac{gp}{mRT}\vec{\nabla}w\frac{\partial \vec{v}}{\partial \eta} - d_3 D_3 + d_3 D + \frac{\partial \mathcal{X}}{\partial t}$$

- evolution $\frac{d\mathcal{X}}{dt}$ is implemented in an approximative way to avoid presence of highly nonlinear terms in equation. Therefore it is evaluated as

$$\frac{d\mathcal{X}}{dt} = \frac{\partial \mathcal{X}}{\partial t} + \vec{v}\vec{\nabla}\mathcal{X} + \dot{\eta}\frac{\partial \mathcal{X}}{\partial \eta}$$

- the advection of d_4 is advected with $d_4 - \mathcal{X}$ and it contains $\vec{\nabla}\mathcal{X}$ term

$$\frac{\partial d_4}{\partial t} = \frac{dd_3}{dt} + \frac{\partial \mathcal{X}}{\partial t} - \vec{v}\vec{\nabla}(d_4 - \mathcal{X}) - \dot{\eta}\frac{\partial (d_4 - \mathcal{X})}{\partial \eta}$$

- evaluation of term with $\vec{\nabla}w$ contains $\vec{\nabla}\mathcal{X}$

$$\vec{\nabla}d_4 = \vec{\nabla}\mathcal{X} - \vec{\nabla}\left(\frac{gp}{mRT}\right)\frac{\partial w}{\partial \eta} - \frac{gp}{mRT}\frac{\partial \vec{\nabla}w}{\partial \eta}$$

Semi-implicit schemes

To control the signals involved in the $\mathcal{M}$ system we traditionally in NWP use the semi-implicit approach with leap-frog time stepping

$$\frac{X^+ - X^-}{2\Delta t} = \mathcal{M}(X)^t - \beta\mathcal{L}^* X^t + \beta\mathcal{L}^* \left(\frac{X^+ + X^-}{2}\right)$$

resp. with two-time level time stepping

$$\frac{X^+ - X^t}{\Delta t} = \mathcal{M}(X)^{t+\frac{\Delta t}{2}} - \beta\mathcal{L}^* X^{t+\frac{\Delta t}{2}} + \beta\mathcal{L}^* \left(\frac{X^+ + X^t}{2}\right)$$

The operator $\mathcal{L}^*$ is linear and it is supposed to include terms which are responsible for:

- gravity modes for hydrostatic models
- gravity and acoustic modes for non-hydrostatic models

The SI scheme is centered and therefore $O(\Delta t^2)$ accurate. It is consistent because

$$\lim_{\Delta t -> 0} \mathcal{L}^* \left(\frac{X^+ + X^-}{2} - X^t\right) = 0$$

Explicit scheme with ALADIN

In model ALADIN the parameter β is represented by the namelist variable NAMDYN/BETADT. With $BETADT = 0$ in namelist, we could run model ALADIN with the explicit leap-frog (3TL) scheme. But this would require time step restricted by the phase speed of sound waves $\Rightarrow \Delta t \approx 3s$ for domain with $dx = 2.5km$.

$$\frac{X^+ - X^-}{2\Delta t} = \mathcal{M}(X)^t$$

I have never seen namelist with $BETADT \neq 1$.

There are three version of SI scheme implemented in the model ALADIN. Three-time level Eulerian scheme (3TL EUL), three-time level semi-Lagrangian (3TL SL) scheme and two-time level semi-Lagrangian scheme (2TL SL). To run one of this schemes following variables have to be set in model namelist

Namelist	Variable	3TL EUL	3TL SL	2TL SL
NAMDYN	NSITER	0	0	0
NAMCT0	LSLAG	F	T	T
NAMCT0	LTWOTL	F	F	T

3TL SI EUL scheme - implementation in the model

- the scheme with the traditional leap frog time marching and Eulerian advection (we suppressed β)

$$\frac{X^+ - X^-}{2\Delta t} = -A(X)^t + \mathcal{M}(X)^t - \mathcal{L}^* X^t + \mathcal{L}^* \left(\frac{X^+ + X^-}{2} \right)$$

- the state X^- is unknown at the beginning of integration. Therefore the following scheme is implemented during the first time step

$$\frac{X^+ - X^t}{\Delta t} = -A(X)^t + \mathcal{M}(X)^t - \mathcal{L}^* X^t + \mathcal{L}^* \left(\frac{X^+ + X^t}{2} \right)$$

$A(X)^t$ - vertical and horizontal advection

3TL SI EUL scheme - implementation in the model

- we introduce the nonlinear residual term

$$\mathcal{R}(X) = \mathcal{M}(X) - \mathcal{L}^*.X$$

- Putting implicit terms on LHS of equation we obtain the scheme as it is implemented into model

$$(I - \Delta t\mathcal{L}^*)X^+ = -A(X)^t + (I + \Delta t\mathcal{L}^*)X^- + 2\Delta t\mathcal{R}(X)^t$$

3TL SI EUL scheme - specificities of d_4 choice

Because $\frac{\partial \mathcal{X}}{\partial t}$ is computed using finite differences then the advection of d_4 is

$$\frac{\partial d_4}{\partial t} = -\vec{v}\vec{\nabla}(d_4 - \mathcal{X}) - \dot{\eta}\frac{\partial d_4 - \mathcal{X}}{\partial \eta} + \frac{\partial \mathcal{X}}{\partial t}.$$

and it requires $\vec{\nabla}\mathcal{X}$ term.

- the estimate $\mathcal{X}^+$ is computed as

$$\tilde{\mathcal{X}}^+ = 2\mathcal{X}^t - \mathcal{X}^-$$

- we evaluate $\frac{\partial \mathcal{X}}{\partial t}$ as

$$\frac{\partial \mathcal{X}}{\partial t} = \frac{\tilde{\mathcal{X}}^+ - \mathcal{X}^-}{2\Delta t}$$

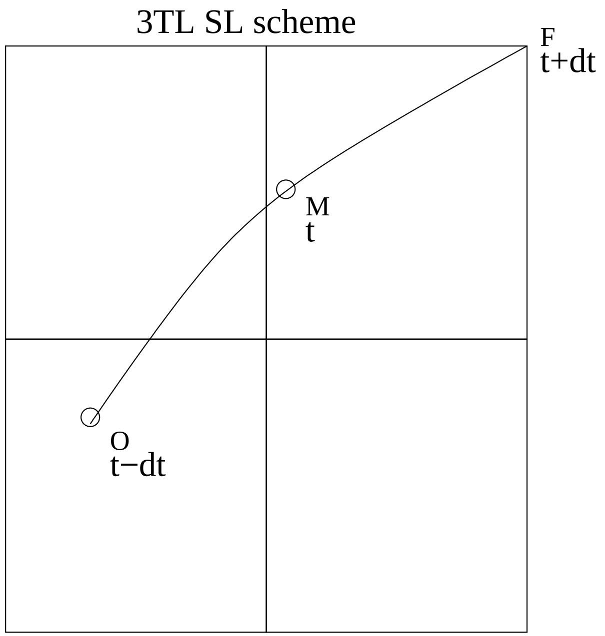
- the estimate $\tilde{\mathcal{X}}^+$ is transformed into spectral space and $\vec{\nabla}\tilde{\mathcal{X}}^+$ is computed. It is used next time step to evaluate $\vec{\nabla}w$ and advection of d_4 variable.

3TL SI SL scheme - implementation in the model

Notation

$$(t + \Delta t) = +$$

$$(t - \Delta t) = -$$



- the scheme with the traditional leap frog time marching and semi-Lagrangian advection (we suppressed β)

$$\frac{X_F^+ - X_O^-}{2\Delta t} = \mathcal{M}(X)_M^t - \mathcal{L}^* X_M^t + \mathcal{L}^* \left(\frac{X_F^+ + X_O^-}{2} \right)$$

- the state X^- is unknown at the beginning of integration. Therefore the following scheme is implemented during the first time step

$$\frac{X_F^+ - X_O^t}{\Delta t} = \mathcal{M}(X)_M^t - \mathcal{L}^* X_M^t + \mathcal{L}^* \left(\frac{X_F^+ + X_O^t}{2} \right)$$

3TL SI SL scheme - implementation in the model

- Putting implicit terms on LHS of equation we obtain

$$(I - \Delta t \mathcal{L}^*) X_F^+ = (I + \Delta t \mathcal{L}^*) X_0^- + 2\Delta t \mathcal{R}(X)_M^t$$

- the M point terms are averaged along the trajectory to avoid interpolations into M point

$$\psi_M = 0.5\,(\psi_F + \psi_O)$$

- Finally it gives the scheme in the form as it is implemented in the model

$$(I - \Delta t \mathcal{L}^*) X_F^+ = (I + \Delta t \mathcal{L}^*) X_0^- + 2\Delta t \frac{\mathcal{R}(X)_O^t + \mathcal{R}(X)_F^t}{2}$$

3TL SI SL - specificities for d_4 choice

Treatment of terms $\vec{\nabla} w$ and $\frac{\partial \mathcal{X}}{\partial t}$ in d_4 equation

- we compute the estimate $\mathcal{X}^+$

$$\tilde{\mathcal{X}}_F^+ = 2\mathcal{X}_M^t - \mathcal{X}_O^- \approx 2\left(\frac{\mathcal{X}_F^t + \mathcal{X}_O^t}{2}\right) - \mathcal{X}_O^-$$

- we evaluate $\frac{\partial \mathcal{X}}{\partial t}$ as

$$\frac{d\mathcal{X}}{dt} \approx \frac{\tilde{\mathcal{X}}_F^+ - \mathcal{X}_O^-}{2\Delta t}$$

- the estimate $\tilde{\mathcal{X}}^+$ is transformed into spectral space and $\vec{\nabla}\tilde{\mathcal{X}}^+$ is computed. It is used next time step to evaluate $\vec{\nabla} w$.

This procedure will be applied from CY29T4 and later in NH-ALADIN. In earlier versions there are four method how to treat term $\vec{\nabla}\mathcal{X}$ and $\frac{d\mathcal{X}}{dt}$ controlled by variable ND4SYS. Please use values ND4SYS=1 or ND4SYS=2 in older model versions. Those methods are equivalent.

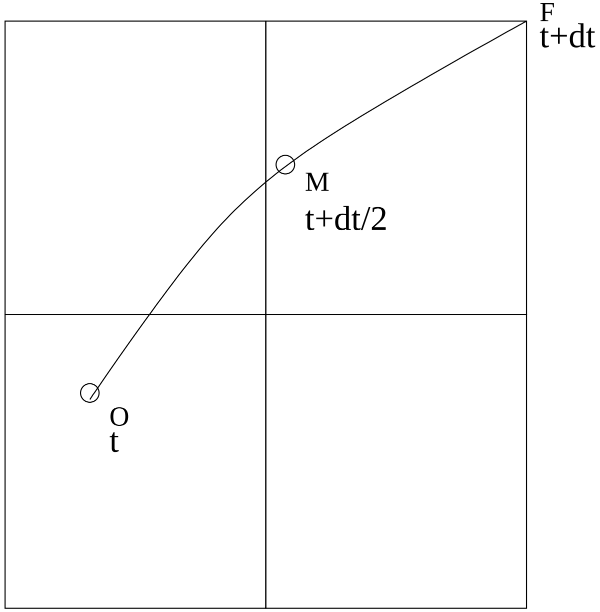
2TL SI SL scheme - implementation in the model

Notation

$$(t+\Delta t) = +$$

$$(t-\Delta t) = -$$

2TL SL scheme



- the 2TL scheme with semi-Lagrangian advection

$$\frac{X_F^+ - X_O^t}{\Delta t} = \mathcal{R}(X)_M^{t+\frac{\Delta t}{2}} + \mathcal{L}^* \left(\frac{X_F^+ + X_O^t}{2} \right)$$

- the state X^- is unknown at the beginning of integration. Therefore the following scheme is implemented during the first time step

$$\frac{X_F^+ - X_O^t}{\Delta t} = \mathcal{R}(X)_M^t + \mathcal{L}^* \left(\frac{X_F^+ + X_O^t}{2} \right)$$

2TL SI SL scheme - implementation in the model

- Putting implicit terms on LHS of equation we obtain

$$(I - \frac{\Delta t}{2}\mathcal{L}^*)X_F^+ = (I + \frac{\Delta t}{2}\mathcal{L}^*)X_0^t + \Delta t \mathcal{R}(X)_M^{t+\frac{\Delta t}{2}}$$

- The term $\mathcal{R}(X)^{t+\frac{\Delta t}{2}}$ is unknown because we do not know the state at time instant $t + \frac{\Delta t}{2}$ (we predict only for times $t + n\Delta t$). The term must be estimated.

2TL SI SL scheme - implementation in the model

There are three methods for $\mathcal{R}(X)^{t+\frac{\Delta t}{2}}$ term extrapolation summarized in the following table. The method is controlled by the namelist keys LSETTLS and LPC_NESC.

NESC approach is only $O(\Delta t)$ and can be used therefore only with iterative schemes where $O(\Delta t^2)$ is restored after first iteration.

		LSETTLS	LPC_NESC
Traditional	$\mathcal{R}_M^{t+\frac{\Delta t}{2}} = \frac{1}{2}\left(\frac{3}{2}\mathcal{R}^t - \frac{1}{2}\mathcal{R}^-\right)_O + \frac{1}{2}\left(\frac{3}{2}\mathcal{R}^t - \frac{1}{2}\mathcal{R}^-\right)_F$	F	F
LSETTLS (Hortal)	$\mathcal{R}_M^{t+\frac{\Delta t}{2}} = \frac{1}{2}\left(2\mathcal{R}^t - \mathcal{R}^-\right)_O + \frac{1}{2}\mathcal{R}_F^t$	T	F
NESC	$\mathcal{R}_M^{t+\frac{\Delta t}{2}} = \frac{1}{2}\left(\mathcal{R}_F^t + \mathcal{R}_O^t\right)$	F	T

2TL SI SL - specificities for d_4 choice

Treatment of terms $\vec{\nabla} w$ and $\frac{\partial \mathcal{X}}{\partial t}$ in d_4 equation

- we compute the estimate $\mathcal{X}^{t+\frac{\Delta t}{2}}$

$$\tilde{\mathcal{X}}_F^+ = 2\mathcal{X}_M^{t+\frac{\Delta t}{2}} - \mathcal{X}_O^t \approx 2\left(\frac{\mathcal{X}_F^{t+\frac{\Delta t}{2}} + \mathcal{X}_O^{t+\frac{\Delta t}{2}}}{2}\right) - \mathcal{X}_O^t$$

  $\mathcal{X}^{t+\frac{\Delta t}{2}}$ is extrapolated using the formula consistent with the extrapolation used to compute $\mathcal{R}(\mathcal{X})^{t+\frac{\Delta t}{2}}$, controlled by LSETTLS and LPC_NESC keys.

- we evaluate $\frac{\partial \mathcal{X}}{\partial t}$ as

$$\frac{d\mathcal{X}}{dt} \approx \frac{\tilde{\mathcal{X}}_F^+ - \mathcal{X}_O^t}{\Delta t}$$

- the estimate $\tilde{\mathcal{X}}^+$ is transformed into spectral space and $\vec{\nabla}\tilde{\mathcal{X}}^+$ is computed. It is used next time step to evaluate $\vec{\nabla} w$.

The choice of semi-implicit linear model $\mathcal{L}^*$ (I)

- traditionally $\mathcal{L}^*$ is obtained by linearization of $\mathcal{M}$ system around the chosen reference state X^*. Due to stability reasons the reference state is
  - isothermal $T^* = const$
  - resting $\vec{v}^* = 0ms^{-1}$
  - horizontally independent $\vec{\nabla}X^* = 0$
  - hydrostatically balanced with ${\pi_s}^* = const$
- The $\mathcal{L}^*$ is involved in the SI solver (Helmholtz equation).

$$(I - \Delta t\mathcal{L}^*)X^+ = RHS(X^t, X^-)$$

  In ALADIN the SI solver is solved in the spectral space and it has to be linear with constant coefficients. This is the reason why X^* must be horizontally independent.

- $\mathcal{L}^*$ is the same for both possible prognostic variables d_3 and d_4 (both variables have the same Helmholtz solver)
- $\mathcal{L}^*$ is a matrix operator dependent only on $T^* = const$ and ${\pi_s}^* = const$.

The key question of SI scheme is how to set $T^* = const$ and ${\pi_s}^* = const$ in order to obtain the stable SI scheme ?

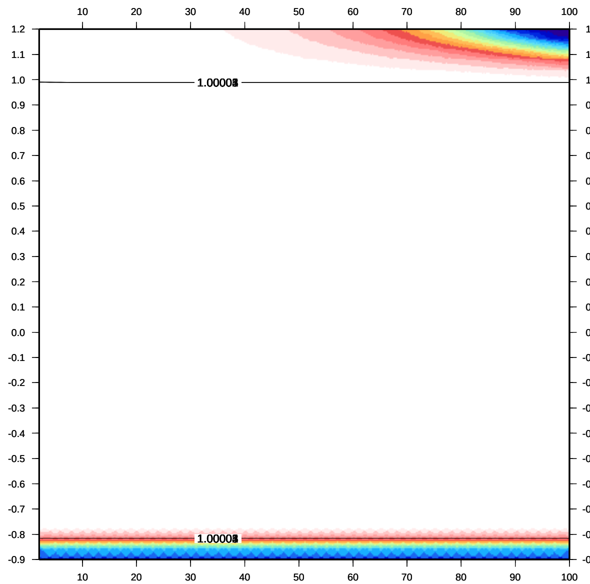
Semi-implicit schemes - stability

The choice of T^* for NH ALADIN

- 3TL SI schemes - T^* shall be set as an average temperature in the domain (based on condition from isothermal analysis $\frac{T^*}{2} < \overline{T} < 2T^*$)
- 2TL SI schemes - unconditionally unstable (in the case of HY model it is stable)

The choice of ${\pi_s}^*$ for NH ALADIN

The model is sensitive to the choice of ${\pi_s}^*$ only if η vertical coordinate is used. The dependency of stability on ${\pi_s}^*$ is eliminated if model is run with $\sigma = \frac{\pi}{\pi_s}$ vertical coordinate. Recommended value for the choice of ${\pi_s}^*$ is an average in the domain or lower.



Dependence of stability on horizontal wavenumber $k = \frac{2\pi}{5000} * n$ and the normalized temperature departure $\frac{\overline{T}-T^*}{T^*}$ for 3TL SI scheme with $\Delta t = 30s$

Semi-implicit schemes - new $\mathcal{L}^*$

The stability of 2TL SI scheme has been retained by Bénard. He proposed to linearize model $\mathcal{M}$ around the two reference temperatures T^* for gravity waves and T_a^* for acoustic waves.

$$
\begin{aligned}
\frac{\partial D}{\partial t} &= -R\mathbf{G}^*\triangle T + RT^*(\mathbf{G}^* - 1)\triangle\widehat{q} - RT^*\triangle ln(\pi_s) - \triangle\phi_s \\
\frac{\partial T}{\partial t} &= -\frac{RT^*}{C_v}\left(\vec{\nabla}D + \widehat{d}\right) \\
\frac{\partial ln(\pi_s)}{\partial t} &= -\mathbf{N}^*D \\
\frac{\partial \widehat{d}}{\partial t} &= -\frac{g^2}{RT_a^*}\mathcal{L}^*\widehat{q} \\
\frac{\partial \widehat{q}}{\partial t} &= -\frac{C_p}{C_v}(\vec{\nabla}D + \widehat{d}) + \mathbf{S}^*D.
\end{aligned}
$$

With new $\mathcal{L}^*$ solver the schemes are stable in the isothermal analysis framework in the range:

- the 2TL SI scheme $\Rightarrow T_a^* < \overline{T} < T^*$.
- the 3TL SI scheme $\Rightarrow \frac{T_a^*}{2} < \overline{T} < 2T^*$.

Semi-implicit schemes - stability in the presence of orography

There are two possible choices for the vertical velocity related prognostic variables d_4 or d_3. The stability of SI time stepping procedure in the presence of orographic forcing is sensitive to the choice of one of this prognostic variables

- NVDVAR=3 - the choice of d_3. The SI scheme is unstable.
- NVDVAR=4 - the choice of d_4. The SI scheme is stable for slight orographic forcing, but for strong orographic forcing with long time steps is unstable.

Semi-implicit schemes - stability in the presence of orography

To ensure the stability of the time stepping also in the presence of orography the more implicit schemes have to be used for the terms responsible for this instability.

The two methods are implemented into NH-ALADIN to ensure more implicit treatment

1. simplified predictor/corrector scheme (NAMDYN/LPC_OLD=.T.)

2. iterative centered implicit scheme (NAMDYN/LPC_FULL=.T.)

At the same time only one of this switches can be set on.

3TL SI schemes - Asselin filter I.

Asselin filter

- We are using the Asselin filter in model ALADIN to suppress the numerical oscillations in the 3TL schemes. It is computed in two steps as:

$$\begin{aligned} timestep(k): \quad & \tilde{X}^t = \alpha X^- + (1-\alpha-\beta)X^t \\ timestep(k+1): \quad & X^- = \tilde{X}^- + \beta X^t \end{aligned} \tag{1}$$

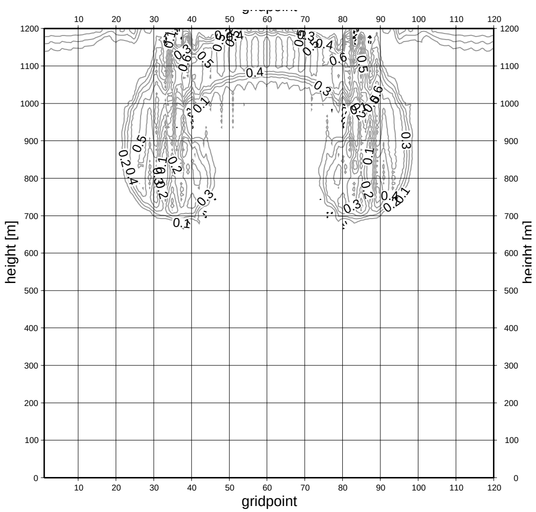
- the following namelist variables must be used whenever LTWOTL=.F. is used

NAMDYN	REPS1=0.05	α for U,V,T,SP
	REPS2=0.05	β for U,V,T,SP
	REPSM1=0.05	α for PD, VD, (AUX)
	REPSM2=0.05	β for PD, VD, (AUX)

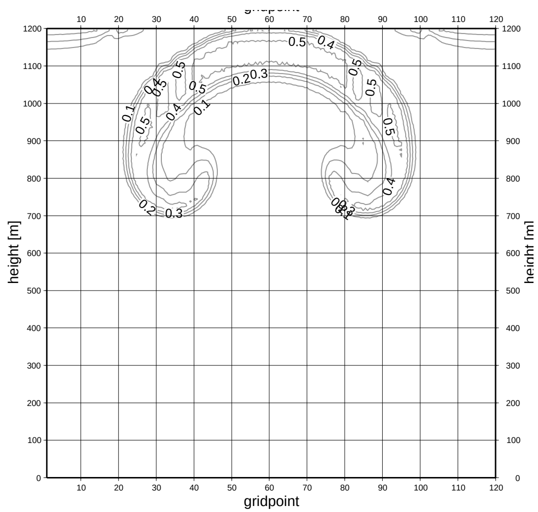
3TL SI schemes - Asselin filter II.

Raising warm bubble at 10m resolution. Runs with $\Delta t = 5s$ with 3TL SL scheme and various values of Asselin filter weights.

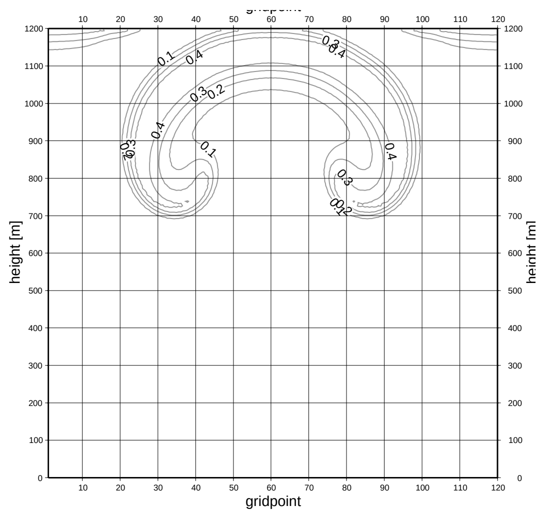
$(\alpha = \beta = 0.0)$



$(\alpha = \beta = 0.05)$



$(\alpha = \beta = 0.5)$



Simplified predictor/corrector scheme - Description I.

We introduce new nonlinear operator $\mathcal{Y}$

$$\frac{X_F^{+(n+1)} - X_O^-}{2\Delta t} = \mathcal{M}(X)_M^t - \mathcal{L}^* X_M^t - \mathcal{Y} X^t + \mathcal{L}^* \left(\frac{X_F^{+(n+1)} + X_O^-}{2} \right) + \left(\frac{\mathcal{Y}(X)_F^{+(n)} + \mathcal{Y}(X)_O^-}{2} \right)$$

$\mathcal{Y}$ is subtracted from explicit residual $\mathcal{M}(X)_M^t - \mathcal{L}^* X_M^t - \mathcal{Y} X^t$ and it is treated implicitly in a iterative manner

$$\left(\frac{\mathcal{Y}(X)_F^{+(n)} + \mathcal{Y}(X)_O^-}{2} \right).$$

The simplified corrector can be used in combination with the 3TL schemes only and with prognostic variable d_3.

Namelist	Variable	3TL EUL	3TL SL
NAMDYN	NSITER	1	1
NAMCT0	LSLAG	F	T
NAMCT0	LPC_OLD	T	T
NAMDYNA	NVDVAR	3	3

Simplified predictor/corrector scheme - Description II.

$$
\begin{aligned}
\frac{du}{dt} &= -\frac{RT}{p}\frac{\partial p}{\partial x} - \left(1 + \frac{1}{m}\frac{\partial(p-\pi)}{\partial \eta}\right)\frac{\partial \phi}{\partial x} + fv \\
\frac{dv}{dt} &= -\frac{RT}{p}\frac{\partial p}{\partial y} - \left(1 + \frac{1}{m}\frac{\partial(p-\pi)}{\partial \eta}\right)\frac{\partial \phi}{\partial y} - fu \\
\frac{dT}{dt} &= -\frac{RT}{C_v}D_3 \\
\frac{\partial q_s}{\partial t} &= -\frac{1}{\pi_s}\int_0^1 \vec{\nabla}\left(m\vec{v}\right) d\eta \\
\frac{dd_3}{dt} &= -\frac{g^2 p}{mRT}\frac{\partial}{\partial \eta}\left(\frac{1}{m}\frac{\partial(p-\pi)}{\partial \eta}\right) + \frac{gp}{mRT}\vec{\nabla} w \frac{\partial \vec{v}}{\partial \eta} - d_3 D_3 + d_3 D \\
\frac{d\widehat{q}}{dt} &= \frac{C_p}{C_v}D_3 - \frac{\omega}{\pi}
\end{aligned}
$$

The $\mathcal{Y}$ is designed to satisfy condition $\mathcal{M} - \mathcal{L}^* - \mathcal{Y} = 0$ for red terms in order to treat elastic D_3 term in a implicit manner.
The $\mathcal{Y}$ contains the following term according to the choice of prognostic variable NVDVAR:

NVDVAR	$\mathcal{Y}$ in T-equation	$\mathcal{Y}$ in $\widehat{q}$-equation
3	$-\frac{RT^*}{C_v}\mathcal{X}$	$-\frac{C_p}{C_v}\mathcal{X}$
4	0	0

Simplified predictor/corrector scheme - implementation into model

The scheme has meaning only in combination with 3TL SL or EUL SI schemes.

$$(I-\Delta t\mathcal{L}^*)X_F^{+(n+1)} = (I+\Delta t\mathcal{L}^*)X_0^- + 2\Delta t\frac{\mathcal{R}(X)_O^t+\mathcal{R}(X)_F^t}{2} + \Delta t\mathcal{Y}(X)_O^- - 2\Delta t\mathcal{Y}(X)_M^t + \Delta t\mathcal{Y}(X)_F^{+(n)}$$

- blue and red terms are precomputed during the SI step (predictor). and then the following iterative procedure is applied

$$(I-\Delta t\mathcal{L}^*)X_F^{+(n+1)} = RHS + \Delta t\mathcal{Y}(X)_F^{+(n)}$$

For the Eulerian scheme the procedure is the same just the content of RHS is different. Simplified PC scheme with d_3 is not sufficiently stable in the presence of strong orographic forcing in the case when long time steps are used. Its properties are the same as the properties of SI scheme with d_4.

From point of view of linear analysis of stability where orography is not considered the stability properties of simplified PC scheme are the same as the properties of SI scheme.

Iterative Centered Implicit schemes I.

The ICI scheme is the iterative approximation of the scheme

$$\frac{X^+ - X^-}{2\Delta t} = \frac{\mathcal{M}(X)^+ + \mathcal{M}(X)^-}{2}$$

This scheme is unconditionally stable in the linear stability analysis context and it slowed down the frequency of waves as

$$\omega_{ICI} = \frac{1}{\Delta t} arctan(\omega \Delta t).$$

Iterations of the scheme are carried out around the existing SI solver.

Iterative Centered Implicit schemes II.

The two versions of ICI schemes were implemented

- 3TL EUL iterative centered implicit scheme
- 2TL SL iterative centered implicit scheme

To run the scheme the following namelist variables have to be turned on

Namelist	Variable	3TL EUL ICI	2TL SL ICI
NAMCT0	LPC_FULL	T	T
NAMCT0	LSLAG	F	T
NAMCT0	LTWOTL	F	T
NAMDYN	NSITER	$>= 1$	$>= 1$

3TL EUL ICI scheme - description I.

- 3TL EUL scheme plays important role in development of dynamics. Its existence allows inter-comparison between SL and EUL version of time stepping procedures.

- it is the iterative approximation of FCI scheme with advection centered in time

$$\frac{X^+ - X^-}{2\Delta t} = A(X)^t + \left(\frac{\mathcal{M}(X)^+ + \mathcal{M}(X)^-}{2}\right)$$

- the first time step

$$\frac{X^+ - X^t}{2\Delta t} = A(X)^t + \left(\frac{\mathcal{M}(X)^+ + \mathcal{M}(X)^t}{2}\right)$$

3TL EUL ICI scheme - description II.

- the n-th iteration of iterative approximation is

$$\frac{X^{+(n+1)} - X^{-}}{2\Delta t} = A(X)^{t} + \left(\frac{\mathcal{M}(X)^{+(n)} + \mathcal{M}(X)^{-}}{2}\right) + \mathcal{L}^{*}\left(\frac{X^{+(n+1)} - X^{+(n)}}{2}\right)$$

- the first iteration (predictor) is the SI EUL scheme

$$\frac{X^{+(0)} - X^{-}}{2\Delta t} = A(X)^{t} + \mathcal{R}(X)^{t} + \mathcal{L}^{*}\left(\frac{X^{+(0)} + X^{-}}{2}\right)$$

3TL EUL ICI scheme - description III.

For the purpose of implementation the corrector is expressed as a correction to the predictor step. Putting implicit terms to the LHS we get:

$$
\begin{aligned}
P: \quad & (I - \Delta t \mathcal{L}^*) X^{t+\Delta t(0)} &=& \; RHS_P \\
C(n): \quad & (I - \Delta t \mathcal{L}^*) X^{t+\Delta t(n)} &=& \; RHS_P \\
& & & + \Delta t \left(\mathcal{R}(X^{t+\Delta t(n-1)}) + \mathcal{R}(X^{t-\Delta t}) - 2\mathcal{R}(X^t) \right)
\end{aligned}
\tag{2}
$$

3TL EUL ICI scheme - advection treatment

Linear advection treatment in leap-frog manner (no-iterations)

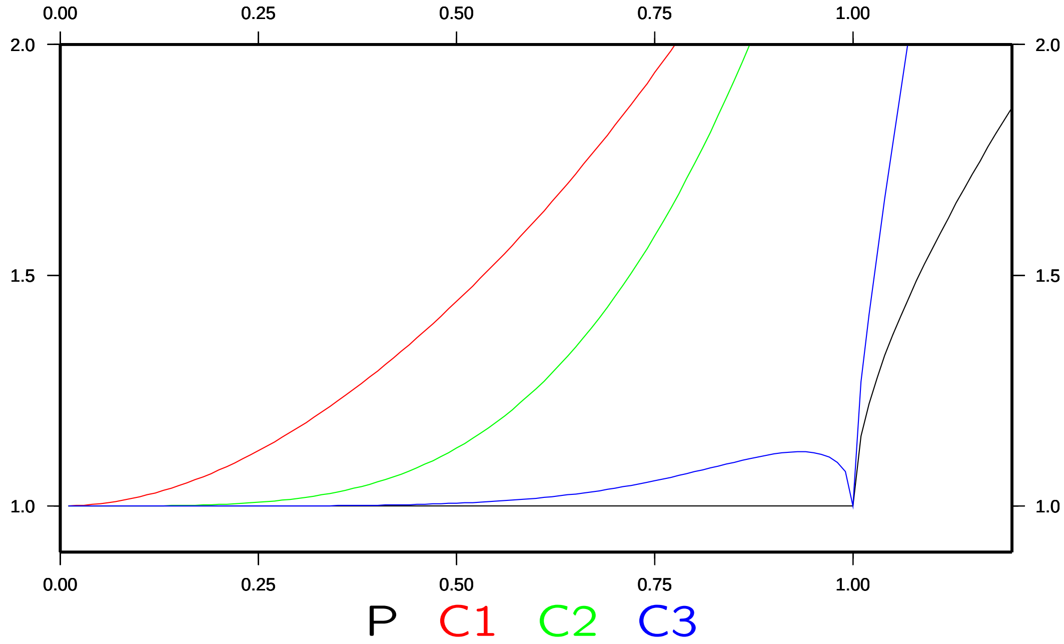
- advection is time centered due to stability consideration

$$\frac{X^{+} - X^{-}}{2\Delta t} = A(X)^{t}$$

Linear advection treatment in 3TL ICI manner with leap-frog predictor

P: $\frac{X^{+(0)} - X^{-}}{2\Delta t} = \bar{U}\frac{\partial X^{t}}{\partial x}$

C: $\frac{X^{+(n+1)} - X^{-}}{2\Delta t} = \bar{U}\frac{\partial}{\partial x}\frac{X^{+(n)} + X^{-}}{2}$



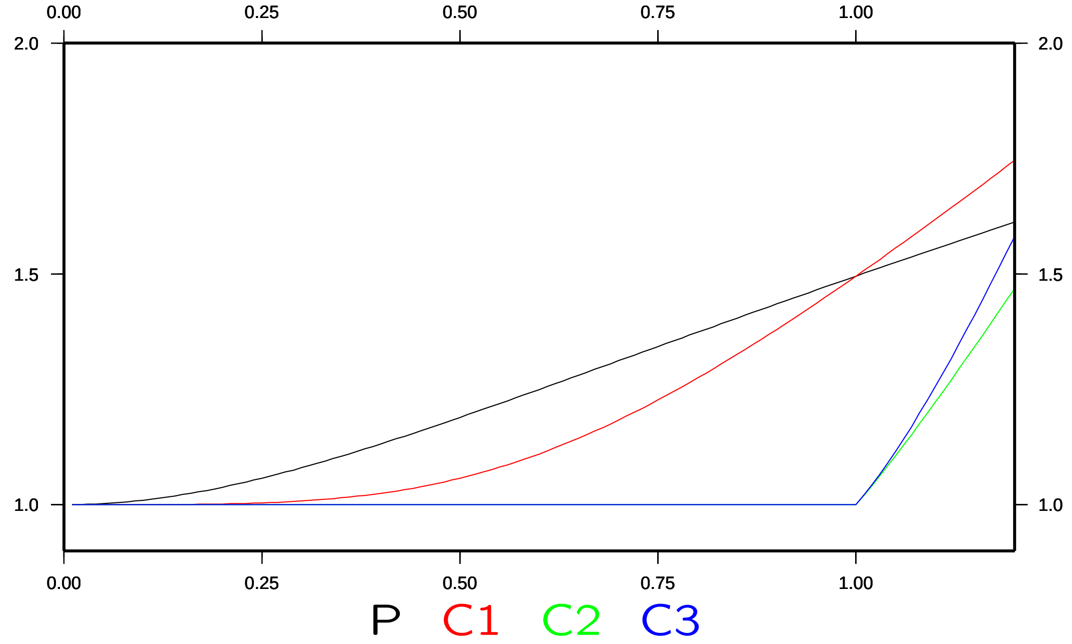
Unstable.

3TL EUL ICI scheme - advection treatment

Linear advection treatment in 2TL ICI manner between + and t.

P: $\frac{X^{+(0)} - X^t}{2\Delta t} = \bar{U}\frac{\partial X^t}{\partial x}$

C: $\frac{X^{+(n+1)} - X^t}{2\Delta t} = \bar{U}\frac{\partial}{\partial x}\frac{X^{+(n)} + X^t}{2}$



We would need to perform at least 2 iterations to get stability for $CFL < 1$ ($CFL = k_{max}\bar{U}\Delta t$)

3TL EUL ICI scheme - specificities of d_4 choice

Treatment of advection, $\vec{\nabla} w$ and $\frac{\partial \mathcal{X}}{\partial t}$ (ND4SYS=5)

- to compute these three terms we compute the estimate $\mathcal{X}^+$ in iterative manner

$$\begin{aligned} P: \quad & \tilde{\mathcal{X}}^+ = 2\mathcal{X}^t - \mathcal{X}^- \\ C(n): \quad & \tilde{\mathcal{X}}^+ = \tilde{\mathcal{X}}^{+(n-1)}. \end{aligned} \tag{3}$$

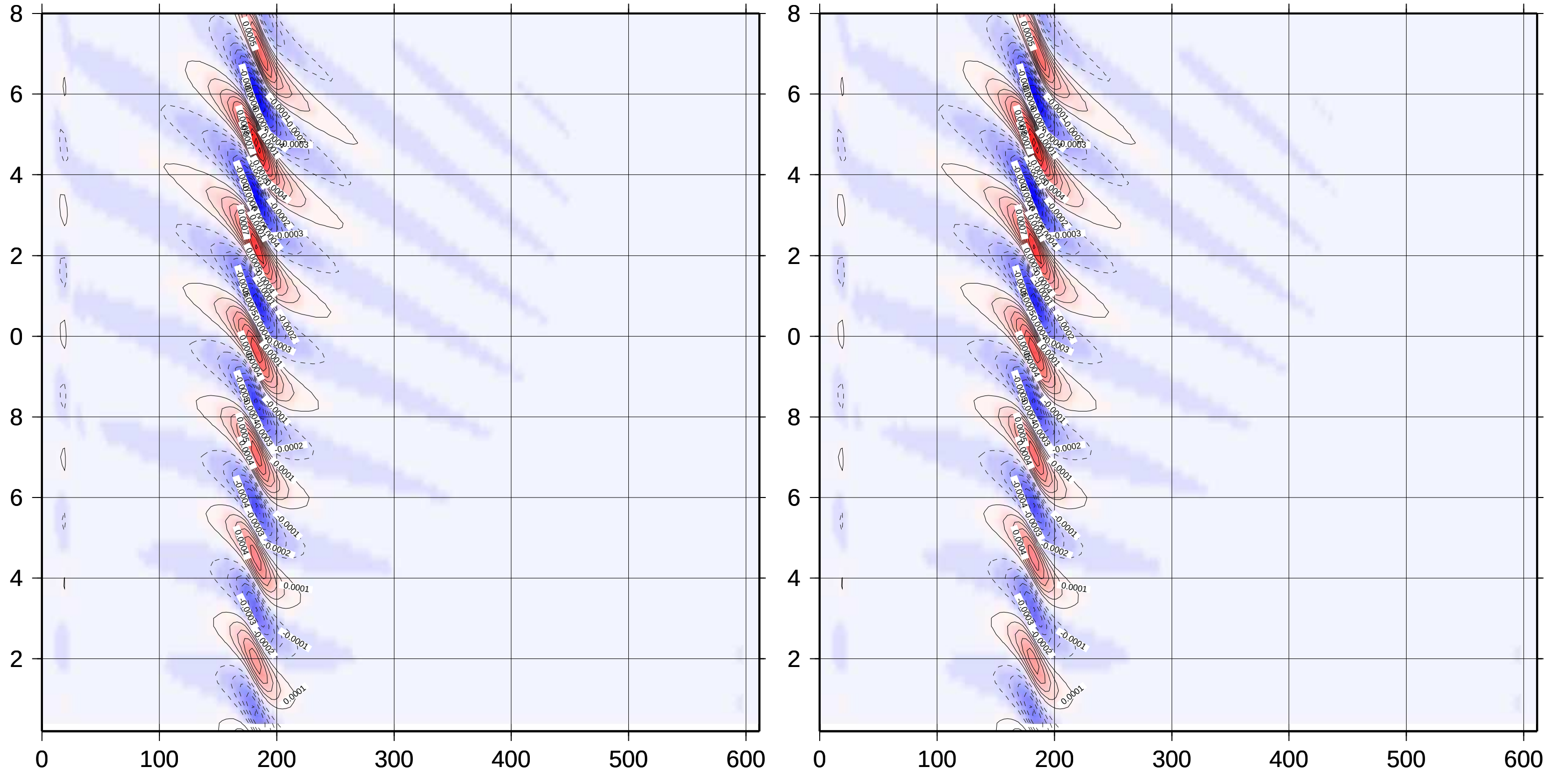
- we evaluate $\frac{\partial \mathcal{X}}{\partial t}$ as

$$\frac{\partial \mathcal{X}}{\partial t} = \frac{\tilde{\mathcal{X}}^+ - \mathcal{X}^-}{2\Delta t}$$

- the estimate $\tilde{\mathcal{X}}^+$ is transformed into spectral space and $\vec{\nabla}\tilde{\mathcal{X}}^+$ is computed. It is used next time step to evaluate $\vec{\nabla} w$ and advection of d_4 variable.

3TL EUL ICI scheme - tests I.

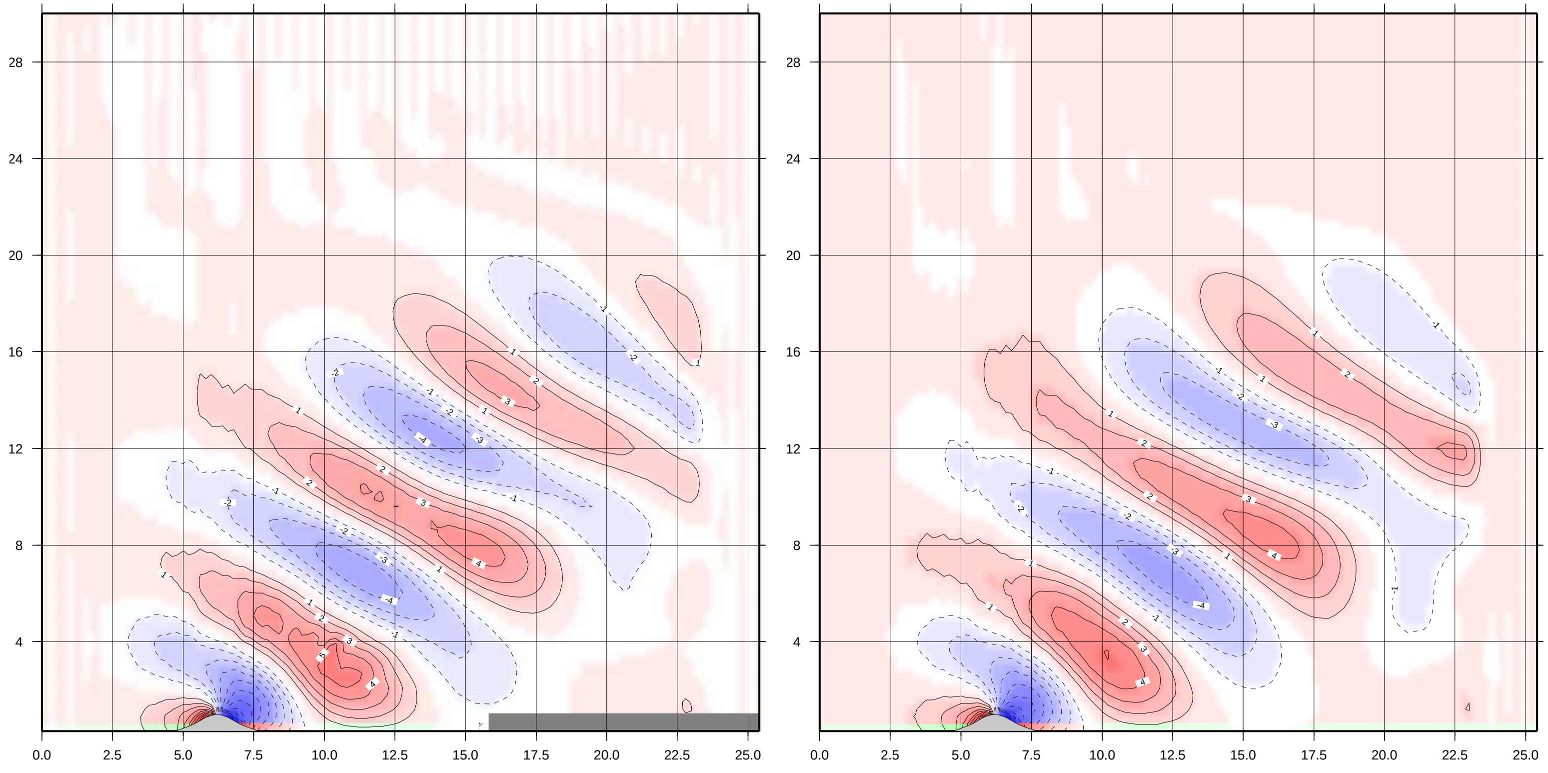
Linear regimes



LIHY: the state after the 500 time steps with $\Delta t = 180s$. The vertical velocity is plotted with step $0.0001ms^{-1}$. The results on the left picture are obtained with the LPC_OLD scheme with 1 iteration and d_3 and on the right with the 3TL EUL ICI scheme with 1 iteration with d_4 prognostic variable.

3TL EUL ICI scheme - tests II.

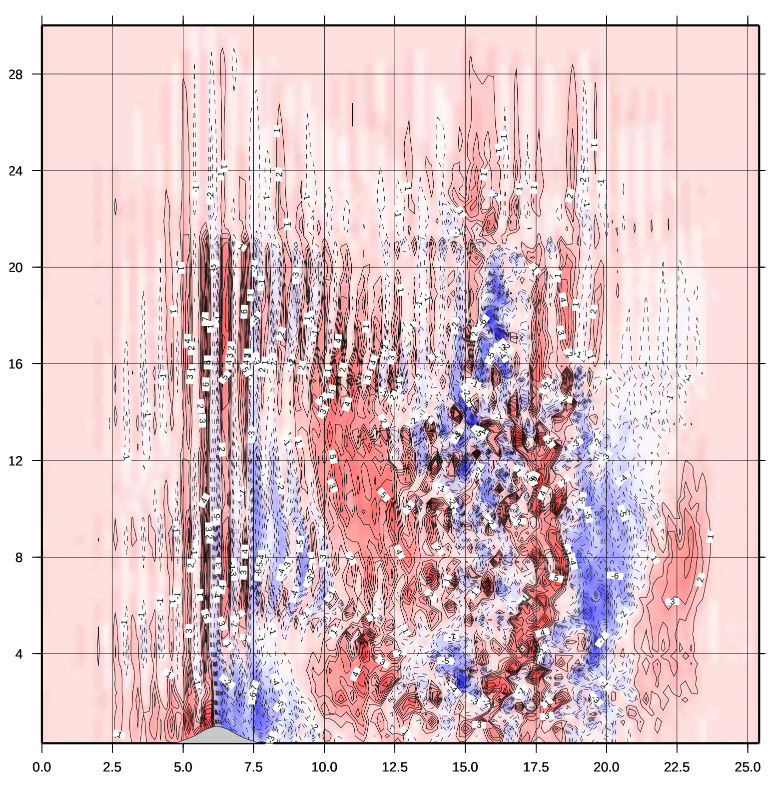
Non-Linear regimes (NVDVAR=3)

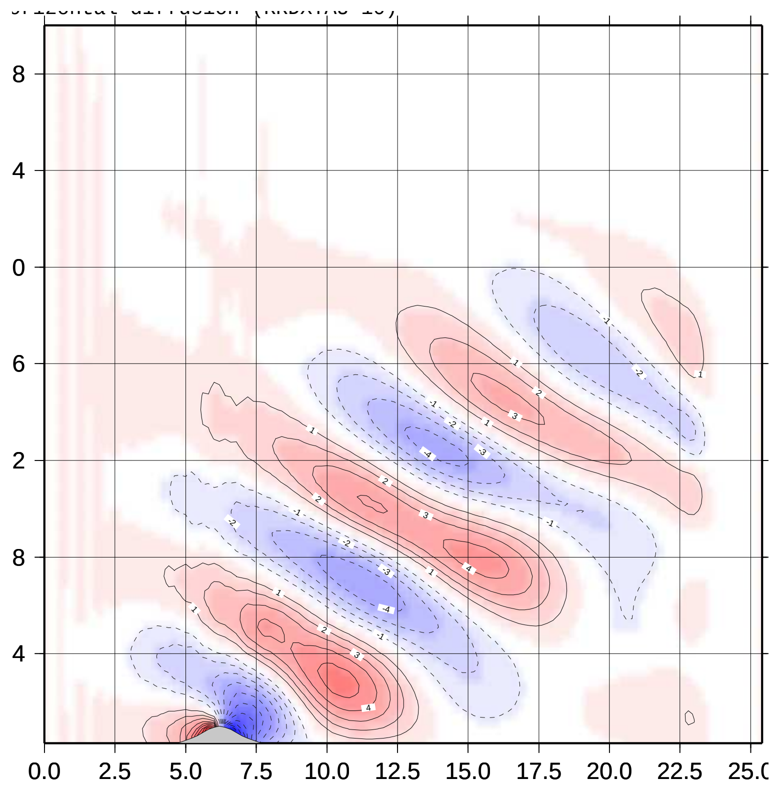


NLNH: the state after the 2000 time steps with $\Delta t = 2.5s$. The vertical velocity is plotted with step $1ms^{-1}$. (left) LPC_OLD scheme with 1 iteration and d_3 (right) 3TL EUL ICI scheme with 3 iteration with d_3. Experiments with 3TL EUL ICI scheme and d_3 with 1 and 2 iterations were unstable.

3TL EUL ICI scheme - tests III.

Non-Linear regimes (NVDVAR=4)





NLNH: the state after the 2000 time steps with $\Delta t = 2.5s$. The vertical velocity is plotted with step $1ms^{-1}$. (left) 3TL EUL ICI scheme with d_4 and no diffusion gives very noisy solution (tested with 1,2 and 3 iterations) (right) 3TL EUL ICI scheme with 1 iteration with d_4 with additional horizontal diffusion.

3TL EUL ICI scheme - stability

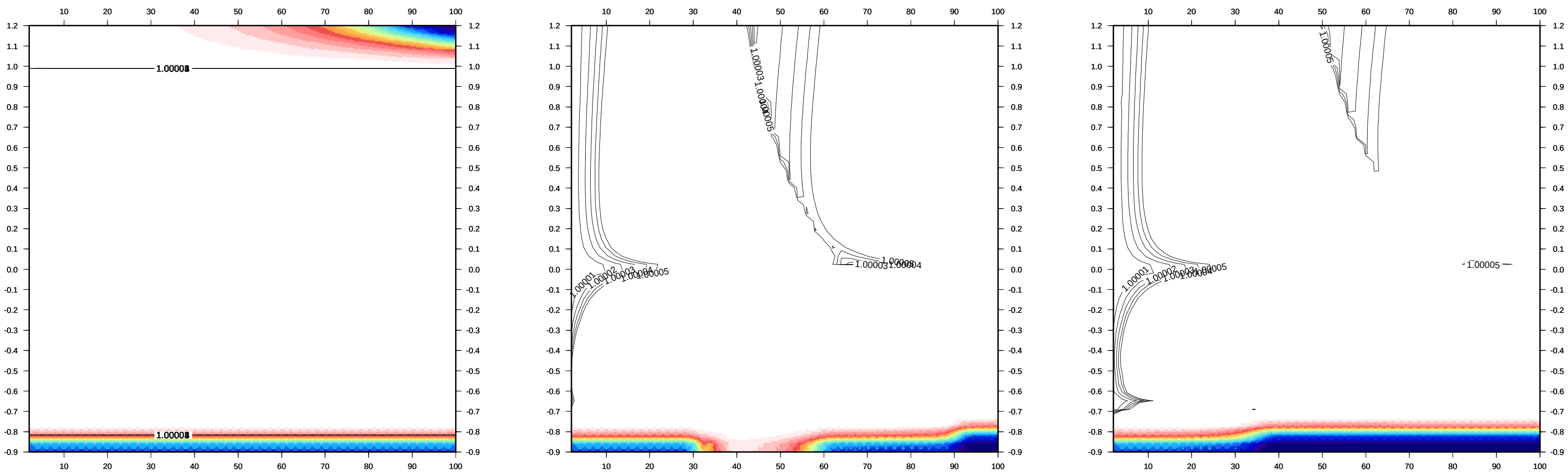
The choice of T^* and T_a^* for NH ALADIN

- 3TL ICI scheme - T^* shall be set as an average temperature in the domain and acoustic temperature T_a^* lower than the lowest temperature in your domain. The stability range from linear analysis of stability is unchanged during the iterative process $\frac{T_a^*}{2} < \overline{T} < 2T^*$.

The choice of $\pi_s{}^*$ for NH ALADIN

- in the case of η coordinate the $\pi_s{}^*$ shall be chosen is an average in the domain or lower

$(iter = 0)$ $(iter = 1)$ $(iter = 2)$



Dependence of stability on horizontal wavenumber $k = \frac{2\pi}{5000} * n$ and the normalized temperature departure $\frac{\overline{T} - T^*}{T^*}$ for 3TL SI scheme with $\Delta t = 30s$

3TL EUL ICI scheme - stability in the presence of orography

There are two possible choices for the vertical velocity related prognostic variables d_4 or d_3. The main difference between them is the stability of SI time stepping procedure in the presence of orographic forcing:

- NVDVAR=3 - 3TL EUL ICI scheme is stable if $NSITER >= 3$.
- NVDVAR=4 - 3TL EUL ICI scheme is stable if $NSITER >= 1$

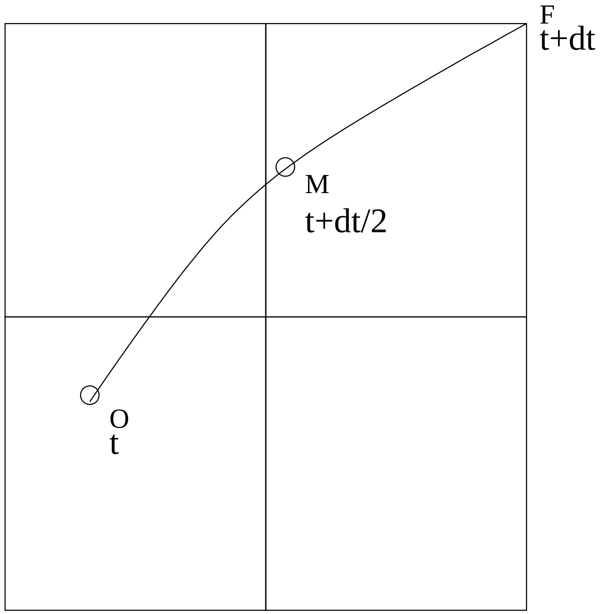
2TL SL ICI scheme - description I.

Notation

$$(t + \Delta t) = +$$

$$(t - \Delta t) = -$$

2TL SL scheme



- the scheme is iterative approximation of Fully Implicit Centered scheme

$$\frac{X_F^+ - X_O^t}{\Delta t} = \frac{\mathcal{M}_F^+ + \mathcal{M}_O^t}{2}$$

- the iterative approximation around existing SI solver gives

$$\frac{X_F^{+(n)} - X_{O(n)}^t}{\Delta t} = \frac{\mathcal{L}^{*+(n)}_F + \mathcal{L}^{*t}_{O(n)}}{2} + \frac{\mathcal{R}_F^{+(n-1)} + \mathcal{R}_{O(n)}^t}{2}$$

2TL SL ICI scheme - description II

The guess $X^{+(0)}$ has to be provided for the iterative procedure. The traditional 2TL SI SL scheme is used to calculate it.

$$\frac{X_F^+ - X_O^t}{\Delta t} = \frac{\mathcal{L}^{*+}_F + \mathcal{L}^{*t}_O}{2} + \mathcal{R}_M^{t+\frac{\Delta t}{2}}$$

with one of the following methods to calculate the nonlinear residual term $\mathcal{R} = (\mathcal{M} - \mathcal{L}^*)$ at time instant $t + \frac{\Delta t}{2}$.

		LSETTLS	LPC_NESC
Traditional	$\mathcal{R}_M^{t+\frac{\Delta t}{2}} = \frac{1}{2}\left(\frac{3}{2}\mathcal{R}^t - \frac{1}{2}\mathcal{R}^-\right)_O + \frac{1}{2}\left(\frac{3}{2}\mathcal{R}^t - \frac{1}{2}\mathcal{R}^-\right)_F$	F	F
LSETTLS (Hortal)	$\mathcal{R}_M^{t+\frac{\Delta t}{2}} = \frac{1}{2}\left(2\mathcal{R}^t - \mathcal{R}^-\right)_O + \frac{1}{2}\mathcal{R}_F^t$	T	F
NESC	$\mathcal{R}_M^{t+\frac{\Delta t}{2}} = \frac{1}{2}\left(\mathcal{R}_F^t + \mathcal{R}_O^t\right)$	F	T

2TL SL ICI - specificities for d_4 choice

Treatment of terms $\vec{\nabla} w$ and $\frac{\partial \mathcal{X}}{\partial t}$ in d_4 equation

- we compute the estimate $\mathcal{X}^{t+\frac{\Delta t}{2}}$

$$\begin{aligned} P: &\quad \tilde{\mathcal{X}}_F^+ = 2\mathcal{X}_M^{t+\frac{\Delta t}{2}} - \mathcal{X}_O^t \approx 2\left(\frac{\mathcal{X}_F^{t+\frac{\Delta t}{2}} + \mathcal{X}_O^{t+\frac{\Delta t}{2}}}{2}\right) - \mathcal{X}_O^t \\ C(n): &\quad \tilde{\mathcal{X}}^+ = \tilde{\mathcal{X}}^{+(n-1)}. \end{aligned} \tag{4}$$

  $\mathcal{X}^{t+\frac{\Delta t}{2}}$ is extrapolated using the formula consistent with the extrapolation used to compute $\mathcal{R}(\mathcal{X})^{t+\frac{\Delta t}{2}}$, controlled by LSETTLS and LPC_NESC keys.

- we evaluate $\frac{\partial \mathcal{X}}{\partial t}$ as

$$\frac{d\mathcal{X}}{dt} \approx \frac{\tilde{\mathcal{X}}_F^+ - \mathcal{X}_O^t}{\Delta t}$$

- the estimate $\tilde{\mathcal{X}}^+$ is transformed into spectral space and $\vec{\nabla}\tilde{\mathcal{X}}^+$ is computed. It is used next time step to evaluate $\vec{\nabla} w$.

2TL SL ICI scheme - stability and convergence

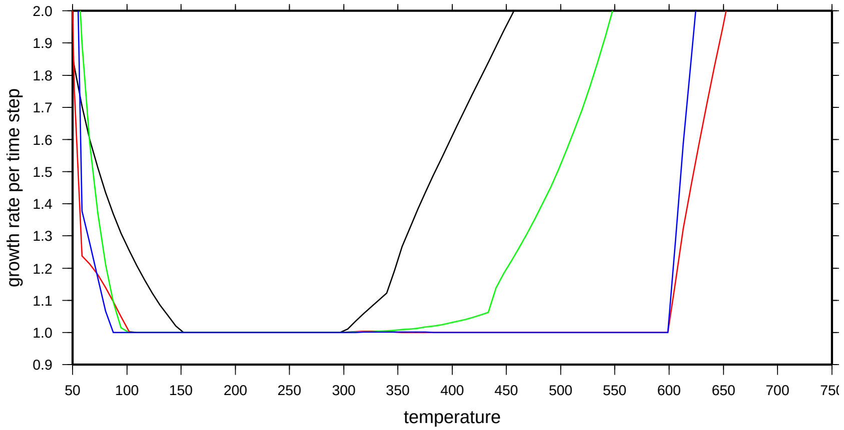
The choice of T^* and T_a^* for NH ALADIN

- 2TL ICI scheme - T^* shall be set higher than any temperature in the the domain and the acoustic temperature T_a^* lower than the lowest temperature in your domain.

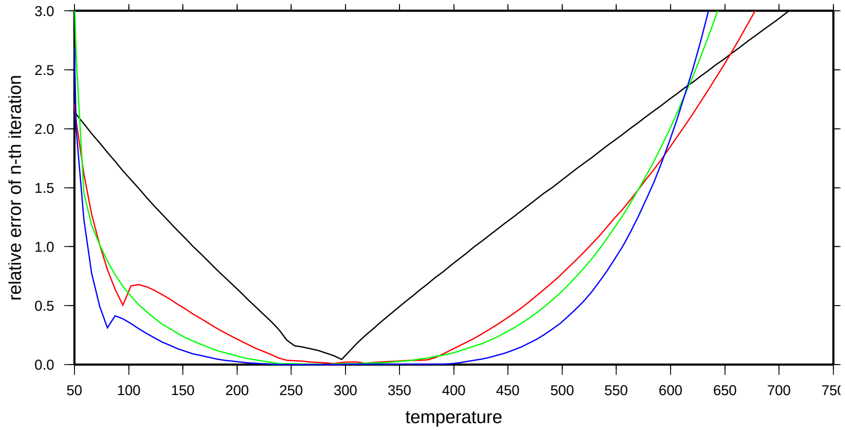
The stability range for temperatures oscillates between the even (0,2,4,..) and odd (1,3,5..) iterations as

Iterations	Stability interval
0,2,4,...	$T_a^* < \overline{T} < T^*$
1,3,5,...	$\frac{T_a^*}{2} < \overline{T} < 2T^*$

Amplification per time step



Convergence towards FIC



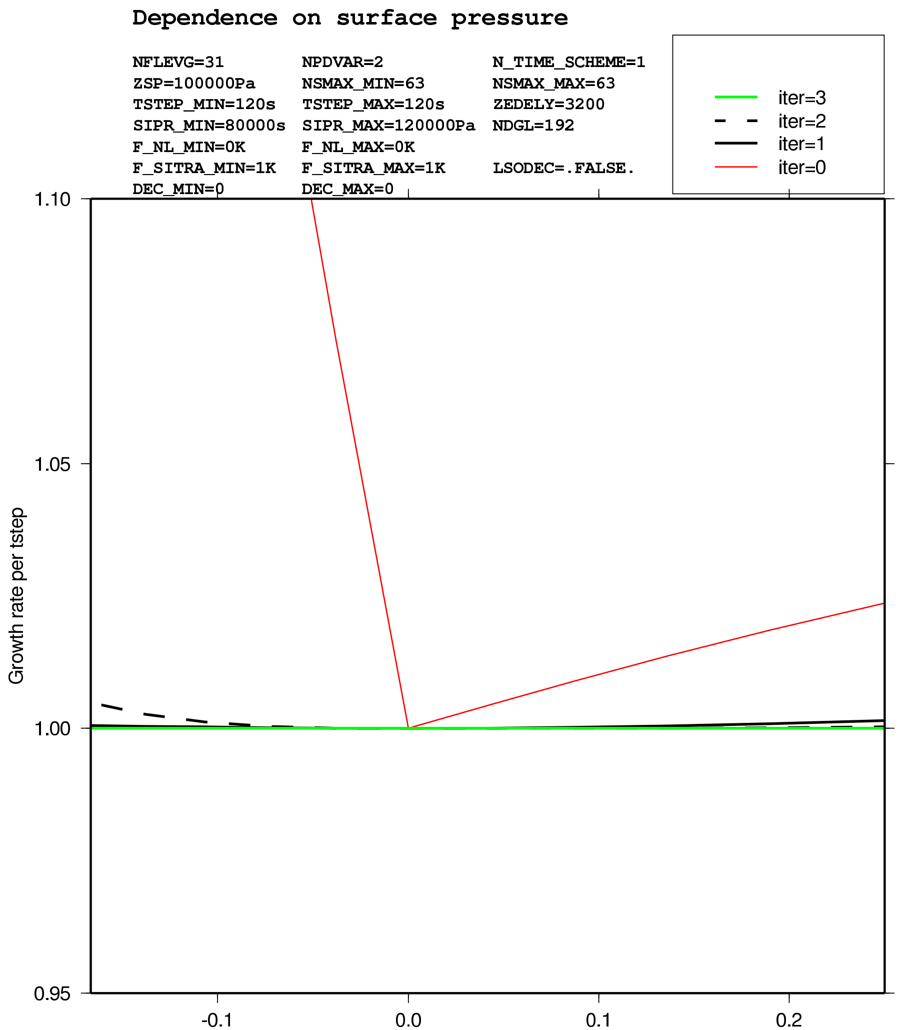
Space continuous analysis in σ coordinate, λ=7.5km,dt=100s,$T^* = 300K$,$T_a^* = 150K$,$\overline{\pi_s} = \pi_s{}^* = 1000hPa$

$$\frac{||X^{t+\Delta t(n)} - X_{FCI}^{t+\Delta t}||}{||X^t||} < \rho \left(A_{ICI(n)} - A_{FCI} \right)$$

2TL SL ICI scheme - stability and convergence

The choice of $\pi_s{}^*$ for NH ALADIN

- 2TL ICI scheme - $\pi_s{}^*$ shall be set as a average value or lower

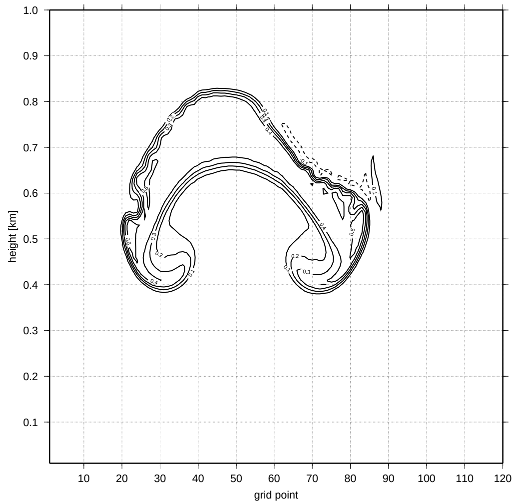


vertically discretized analysis with 31 η levels,
λ=9.6km,dt=120s,$T^* = T_a^* = \overline{T} = 300K$,$\overline{\pi_s} = 1000hPa$

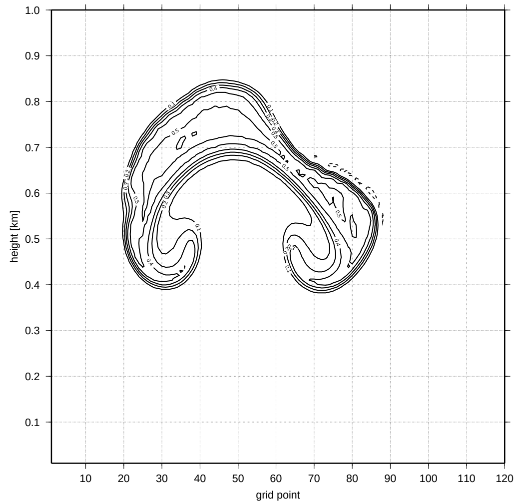
2TL ICI SL - the accuracy of different extrapolations

We have tested the accuracy of the scheme on time evolution of warm and cold bubble. Experiments run with d_4 prognostic variable. The comparison of extrapolation computation.

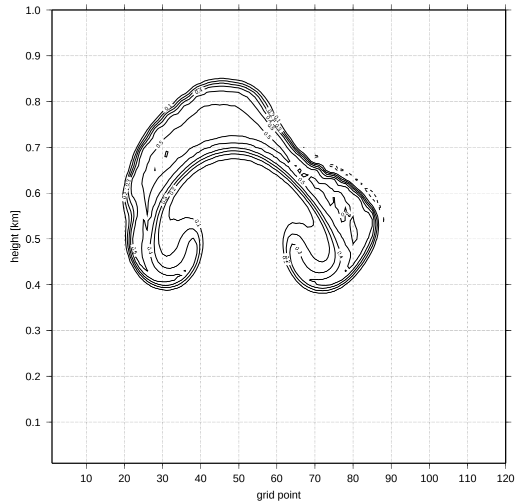
2TL SI NESC



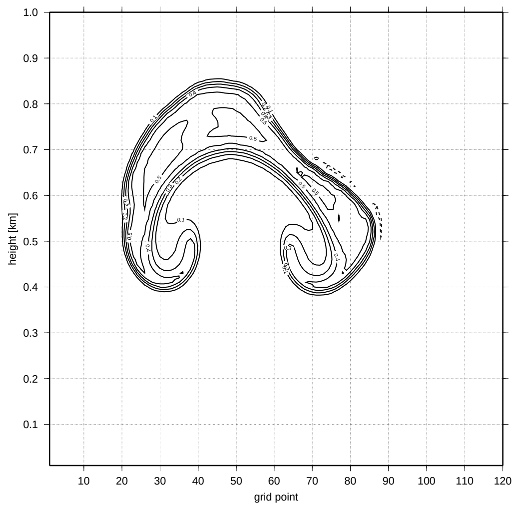
2TL SI LSETTLS



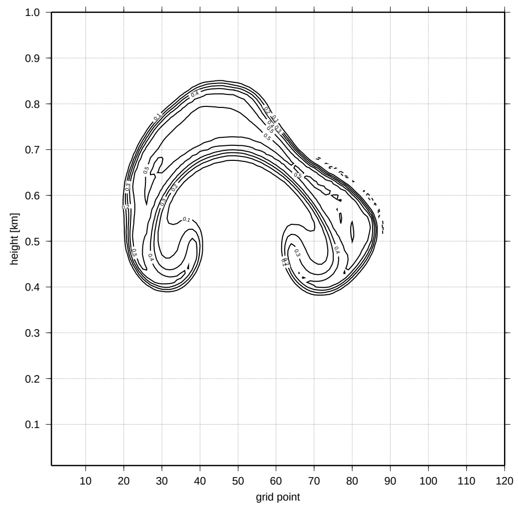
2TL SI Robert



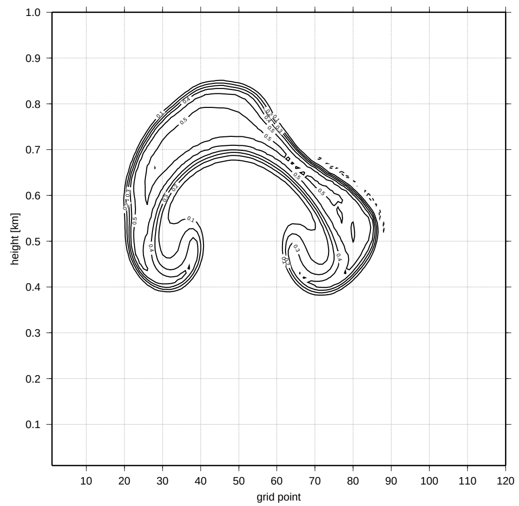
2TL PC NESC



2TL PC LSETTLS



2TL PC Robert



2TL SL ICI - trajectories recomputation

During the development we implemented ICI scheme without trajectories recomputation. The trajectories were computed during the predictor stage using the $O(\Delta t^2)$ method. The idealized experiments experiments showed that:

- it is very difficult to find a positive impact of trajectories recomputation (although the positive impact has been already reported in literature)
- to profit from this approach, the dataflow of time stepping would have to be significantly re-designed in the spirit of LPC_OLD
- the tests were performed with d_3 variable (d_4) was not available.
- the greatest differences has been found with ALPIA test case

2TL SL ICI scheme - trajectories recomputation

Trajectories (position of O point) are recomputed each iteration as

$$\vec{OF}(0) = \Delta t \vec{v}_M^{t+\frac{\Delta t}{2}}$$

1. Predictor: wind is approximated consistently with the calculation of nonlinear residual

Traditional	$\vec{v}_M^{t+\frac{\Delta t}{2}} = \frac{1}{2}\left(\frac{3}{2}\vec{v}^t - \frac{1}{2}\vec{v}^-\right)_O + \frac{1}{2}\left(\frac{3}{2}\vec{v}^t - \frac{1}{2}\vec{v}^-\right)_F$
LSETTLS (Hortal)	$\vec{v}_M^{t+\frac{\Delta t}{2}} = \frac{1}{2}\left(2\vec{v}^t - \vec{v}^-\right)_O + \frac{1}{2}\vec{v}_F^t$
NESC	$\vec{v}_M^{t+\frac{\Delta t}{2}} = \frac{1}{2}\left(\vec{v}_F^t + \vec{v}_O^t\right)$

2. N-th iteration: centered iterative approach

$$\vec{OF}(n) = \Delta t \frac{\vec{v}_F^{+(n-1)} + \vec{v}_{O(n)}^t}{2}$$

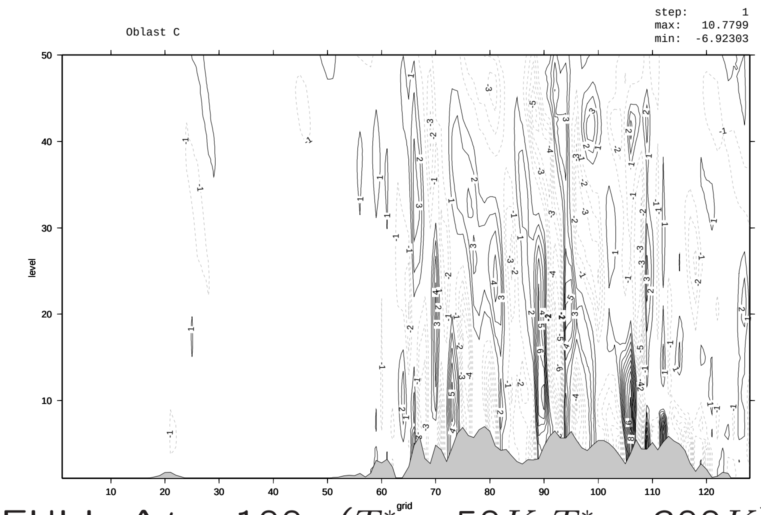
2TL ICI SL schemes - trajectories recomputation

REF Eulerian $\Delta t = 10s$ ($T_a^* = 50K, T^* = 300K$)

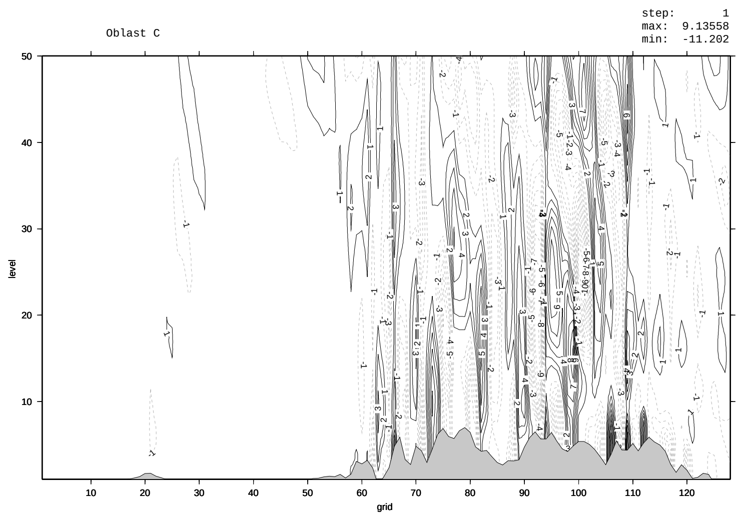
dx=2.5km

$T_s = 300K$

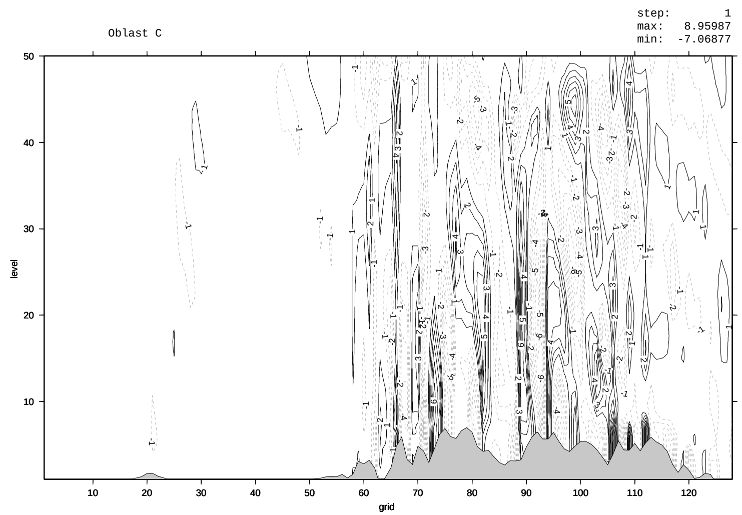
$T_t \approx 50K$



FULL $\Delta t = 100s$ ($T_a^* = 50K, T^* = 300K$)

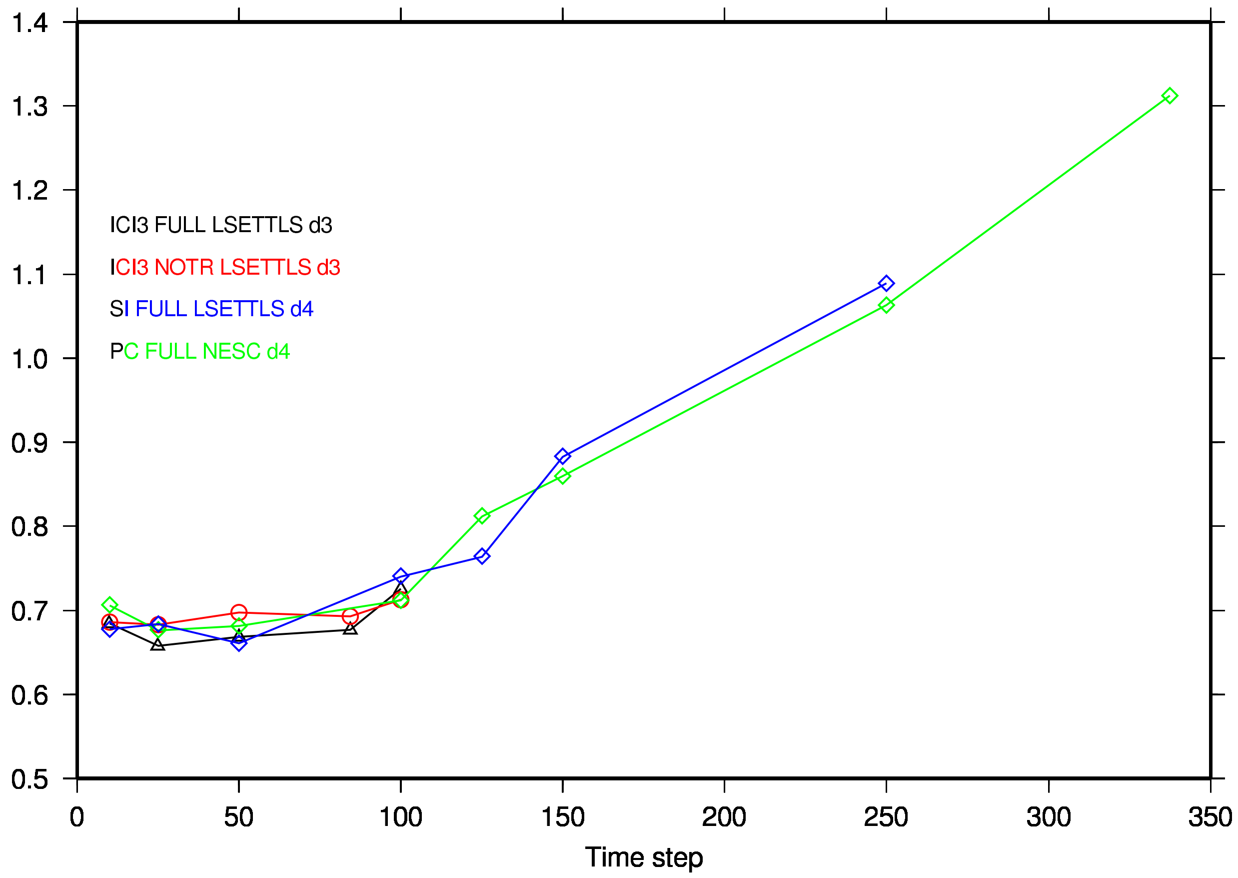


NOTR $\Delta t = 100s$ ($T_a^* = 50K, T^* = 300K$)



2TL ICI SL scheme - recalculations of trajectories

Stdev comparing to REF Eulerian experiment with $\Delta t = 10s$.

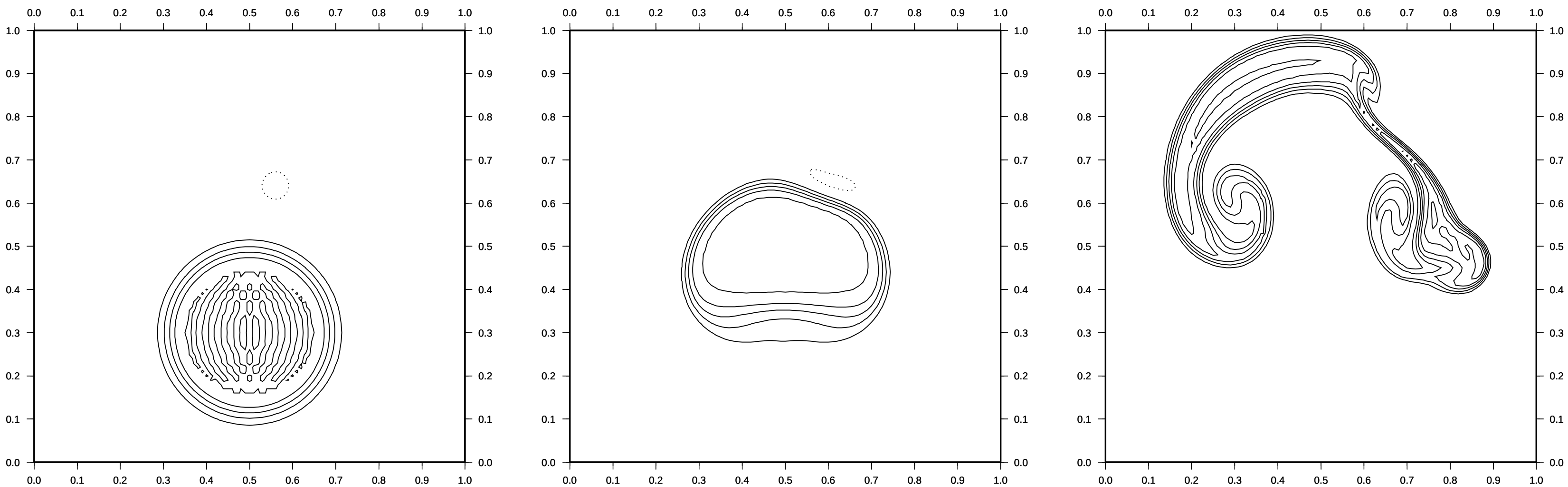


2TL ICI SL - accuracy with Warm and Cold Bubble

- neutral stratification with θ_0=300K, warm and cold bubble - Robert (1993)
- isothermal layer at the top, dx=10m, dz $\approx$ 10m
- reference run Δt=0.2s, 2TL ICI(3), $T^* = 300K$, $T_a^* = 150K$

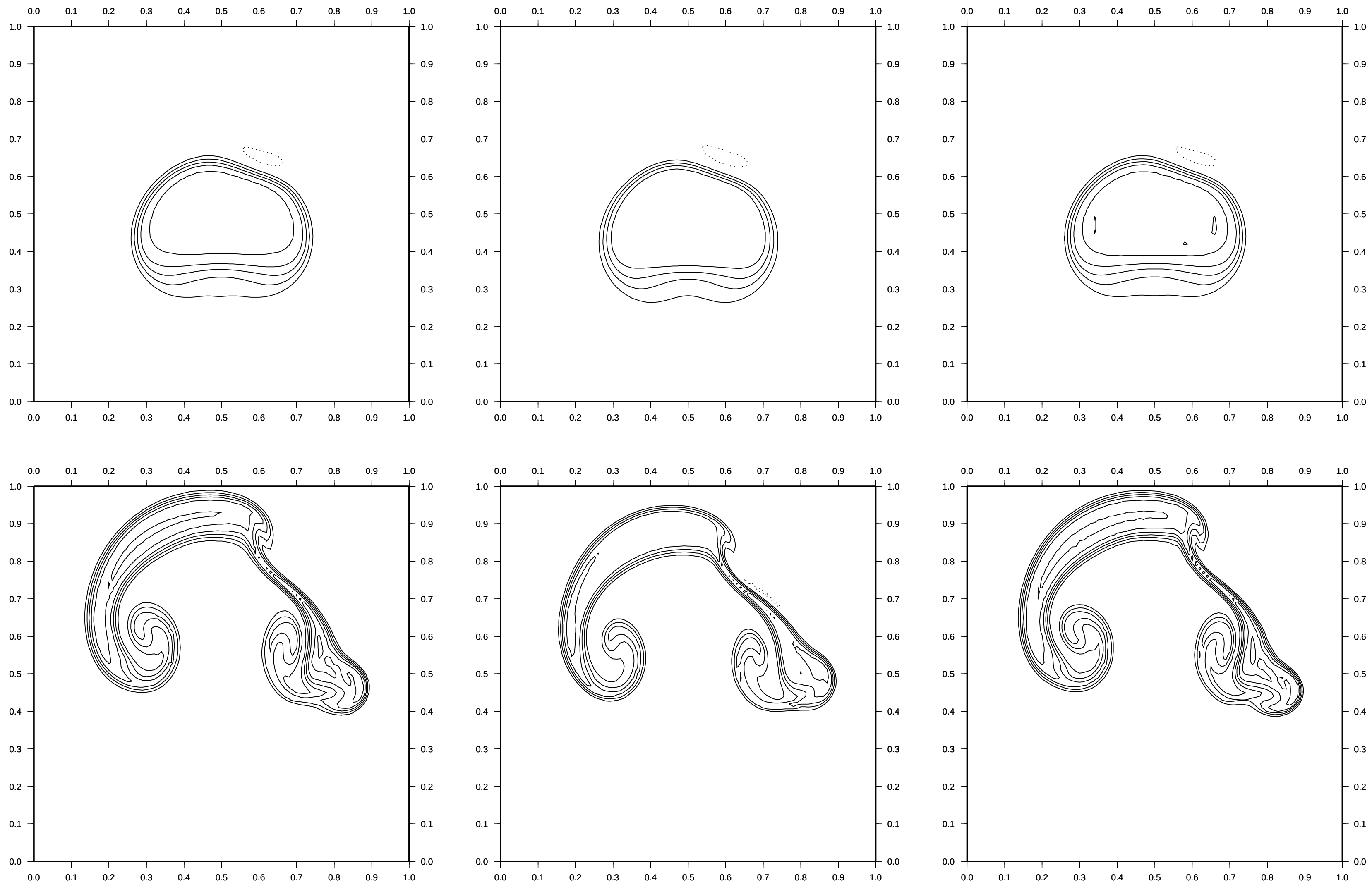
$\theta' = \theta - \theta_0$ (contours $0.1K$)

t=0s t=4min t=9min



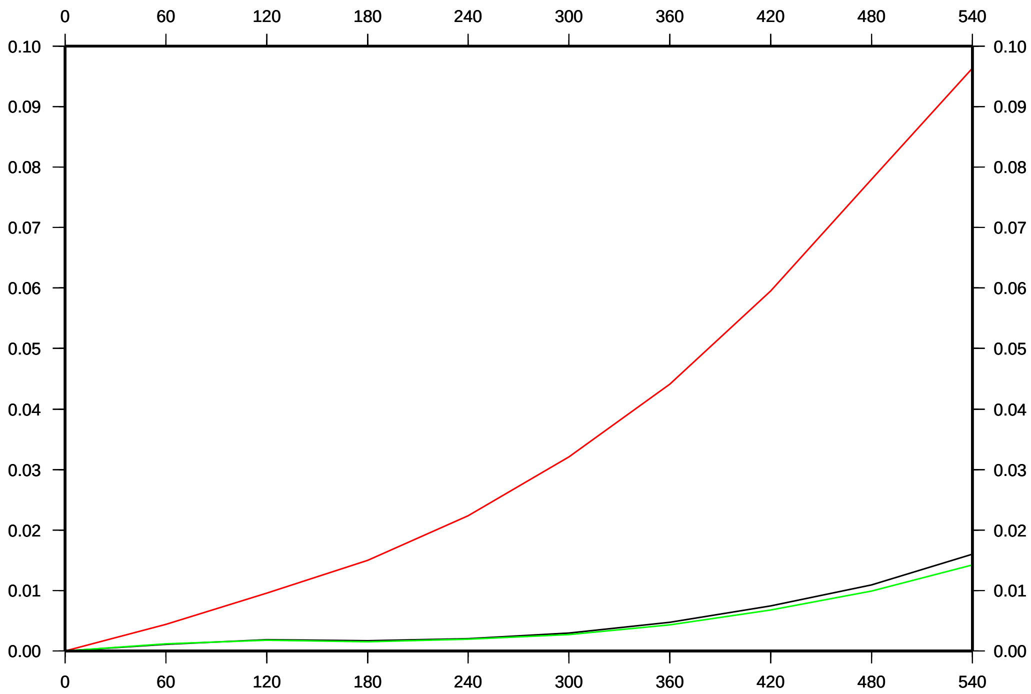
2TL ICI SL - accuracy with Warm and Cold Bubble

2TL ICI(3) $\Delta t = 0.2s$ 2TL ICI(0) $\Delta t = 5s$ 2TL ICI(1) $\Delta t = 5s$



2TL ICI SL - accuracy with Warm and Cold Bubble

Evolution of standard deviation of $\theta_{REF} - \theta_{ICI}$ during integration

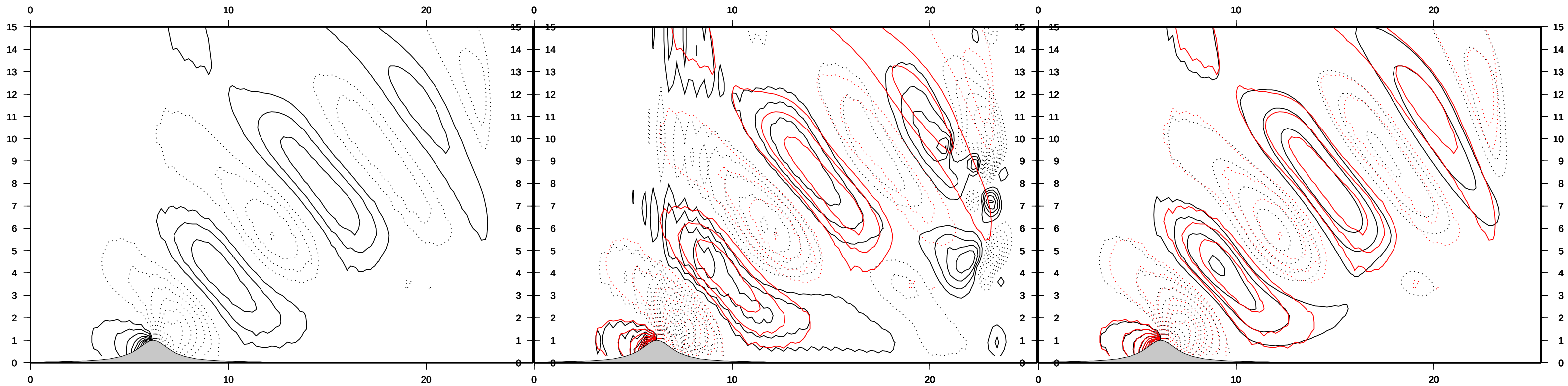


2TL ICI(0)	2TL ICI(1)	2TL ICI(7)

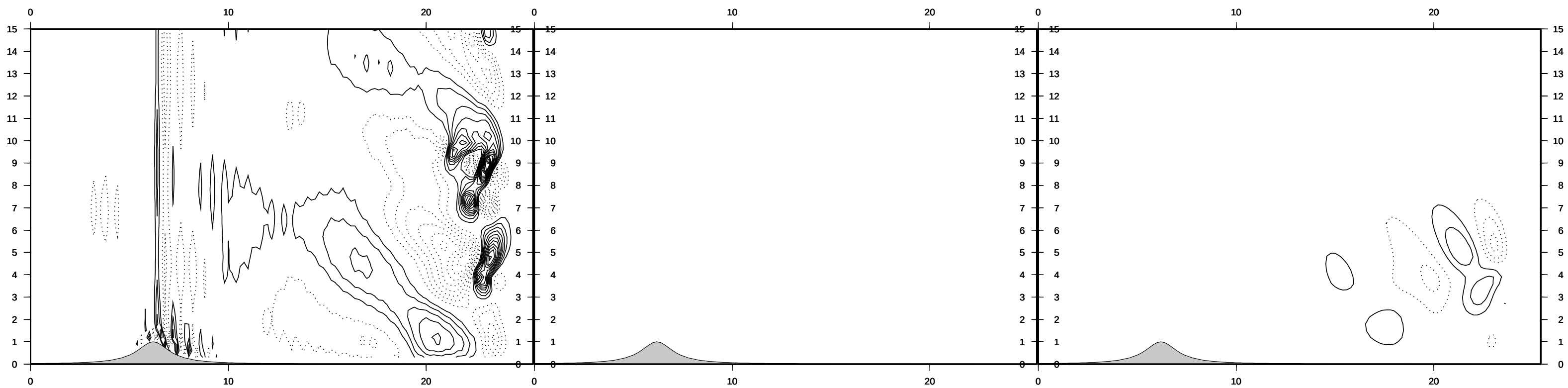
2TL ICI SL - convergence in the presence of steep mountain

NLNH regime simulation with $CFL = 2$ compared with the simulation at the same resolution but run with $CFL << 1$

reference | Diff (iter1-iter0) | Diff (iter2-iter1)



iter=0 | iter=1 | iter=2



Conclusions

- the traditional SI schemes are not sufficiently stable in the presence of strong orographic forcing and the iterative schemes have to be exploited

- the most appropriate available scheme for NH ALADIN is 2TL SL ICI scheme with direct spectral solver

- with the prognostic variable d_4 one iteration is sufficient to get stable and robust time stepping procedure for elastic compressible dynamic cast in hybrid mass coordinate

- the ICI scheme (2TL and 3TL) is convergent towards the unique solution represented by FIC scheme