ALATNET 2002 seminar

Seminar on Numerical Methods and NWP applications

Numerical analysis

part 1: Introduction
part 2: Historical aspects
part 3: Numerical schemes
part 4: Non-linearity

by
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Kranjska Gora, 27 May 2002

The purpose of numerical techniques:

- to solve non-algebraic equation
like: $x=e^x$
if respecting some rules not problematic

- to numerically integrate
not problematic

- to numerically differentiate
in many cases rather problematic.

Numerical methods for
differential equations
when:

- these are “difficult”.

or when

- these are analytically “non-solvable”.

Three main methods:

- finite differences

☺: direct, clearly understandable

☹: not exact approximation of derivatives

- finite elements

☺: mainly for problems with special geometries (in ingeneering)

- spectral

☺: exact computation of derivatives

☹: not appropriate for diabatic effects, thus: often from spectral into physical space and back (with reverse FFT and direct FFT).

How efficient is numerical vs. analytical solution?

As an example: a Bernoulli type equation (time dependant acceleration and friction):

$$\frac{dv}{dt} = a\frac{v}{t} - k\sqrt{v}$$

(for example a=4, k=2)

Analitically: dividing by $\sqrt{v}$, introducing a new variable $\sqrt{v}$, transforms it into a linear equation, being analytically "solvable".

$$\frac{dv}{dt} = 4\frac{v}{t} - 2\sqrt{v}$$

It has a stationary point at v=0!
So not every initial condition is applicable for description of the time development!

Start from v=0 at (any) time means stagnation at this starting point.

For dynamic development one must choose
an initial condition other than
"the most natural" initial point (0, 0)!

For example a condition: v(t=1) = 1.

Such an equation:

$$\frac{dv}{dt} = 4\frac{v}{t} - 2\sqrt{v}$$

has a general solution

$v = t^4(1/t - C)^2$.

The solution of the actual problem depends on initial condition.

$$\frac{dv}{dt} = 4\frac{v}{t} - 2\sqrt{v}$$

A general solution

$v = t^4(1/t - C)^2$

has, with v(t=1)=1

two equally valid solutions

(due to the dependence on C^2)

$v = t^2$ and $v = t^2(1-2t)^2$.

Is a numerical method
capable to reproduce
both:

$$v = t^2$$

and

$$v = t^2(1/t + 2)^2$$?

A test with
finite differences approach.

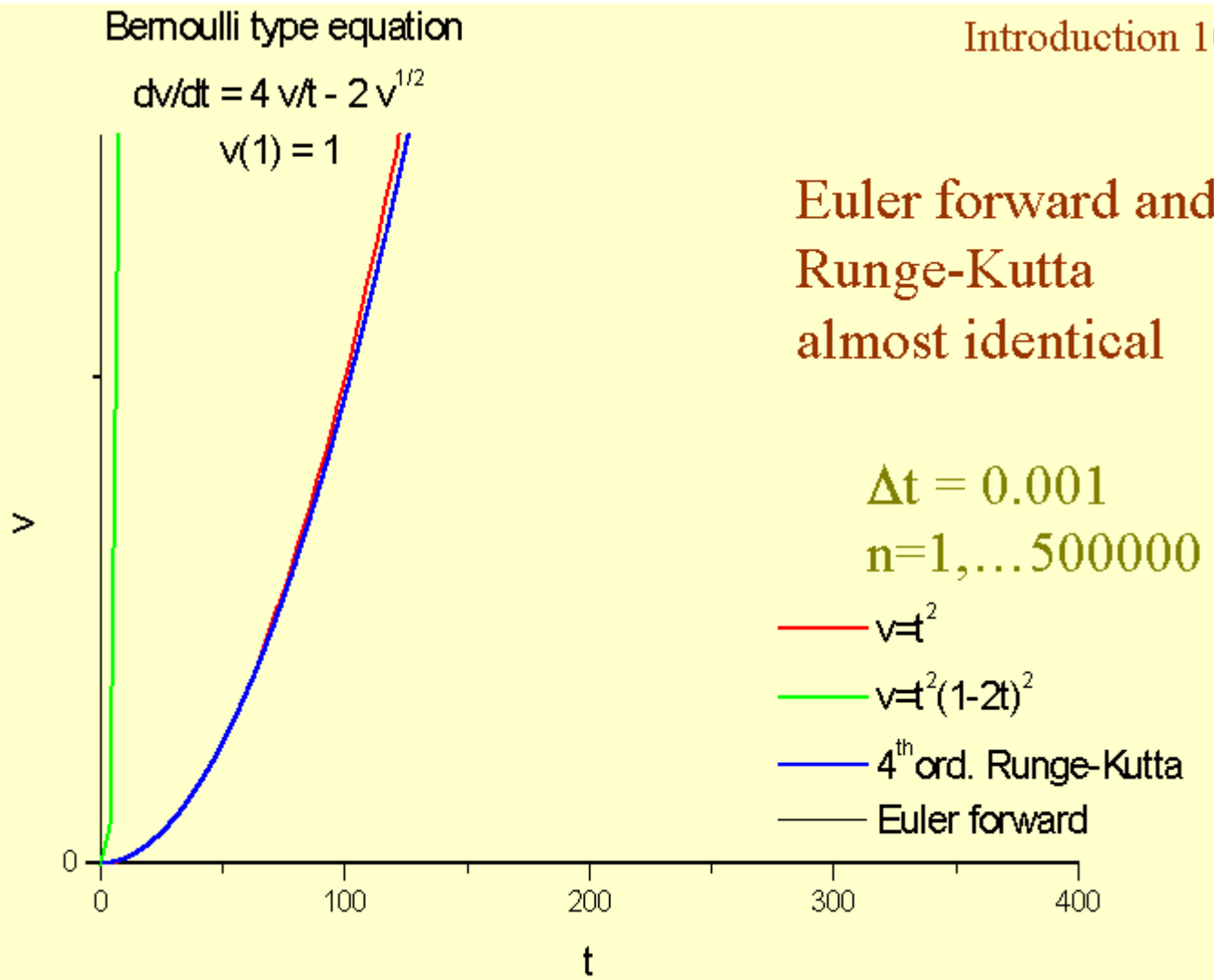
The simplest choice:
Euler forward scheme

$$t(n+1) = t(n) + \Delta t$$

$$V(n+1) = V(n) + \left[4\frac{V(n)}{t(n)} - 2\sqrt{V(n)} \right] \Delta t$$

with initial condition V(n=1) = 1.

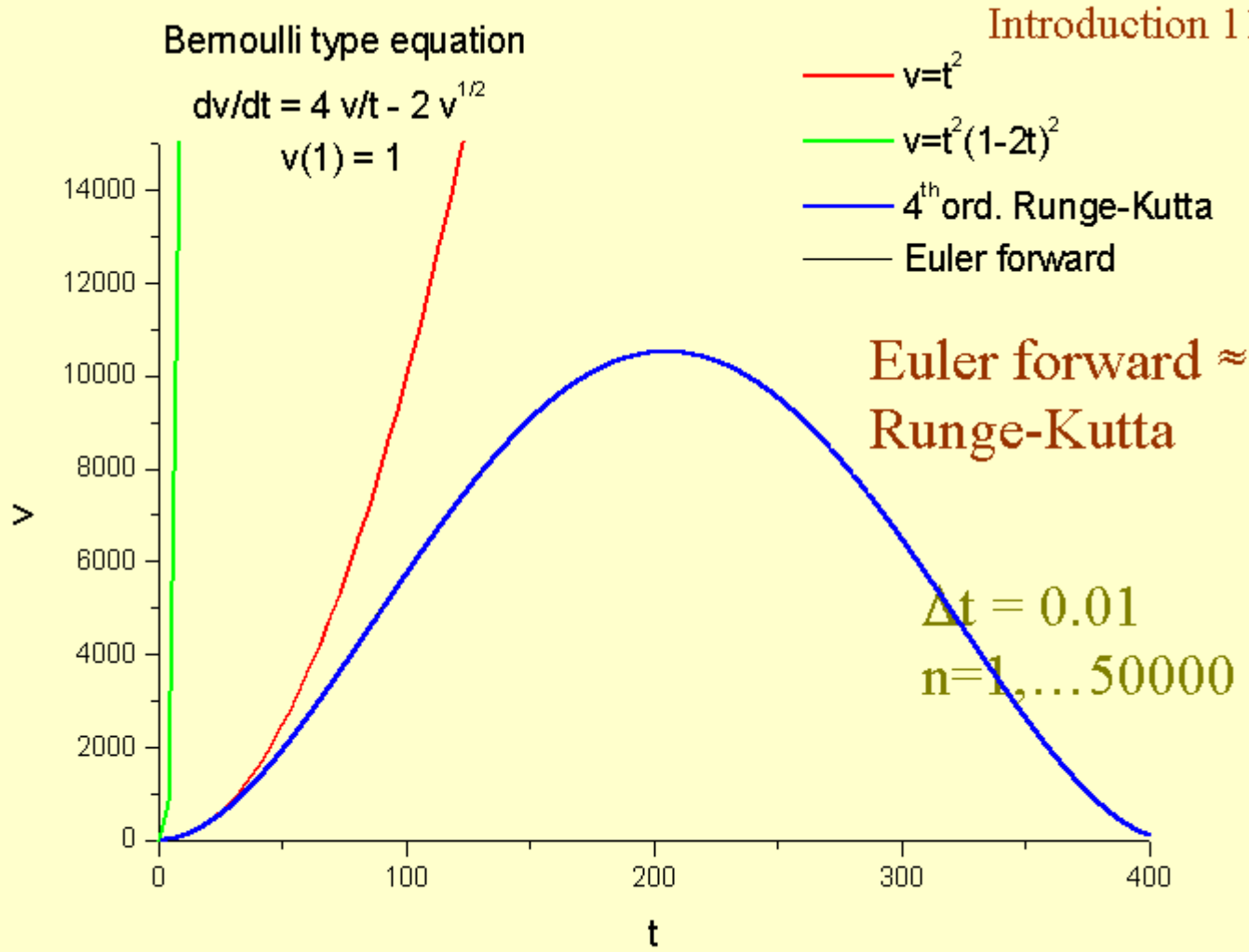
If we do not "trust" such a simple scheme, we may apply a much more complicated and "accurate"one, for example:
the 4th order Runge-Kutta



Euler forward and Runge-Kutta almost identical

$\Delta t = 0.001$

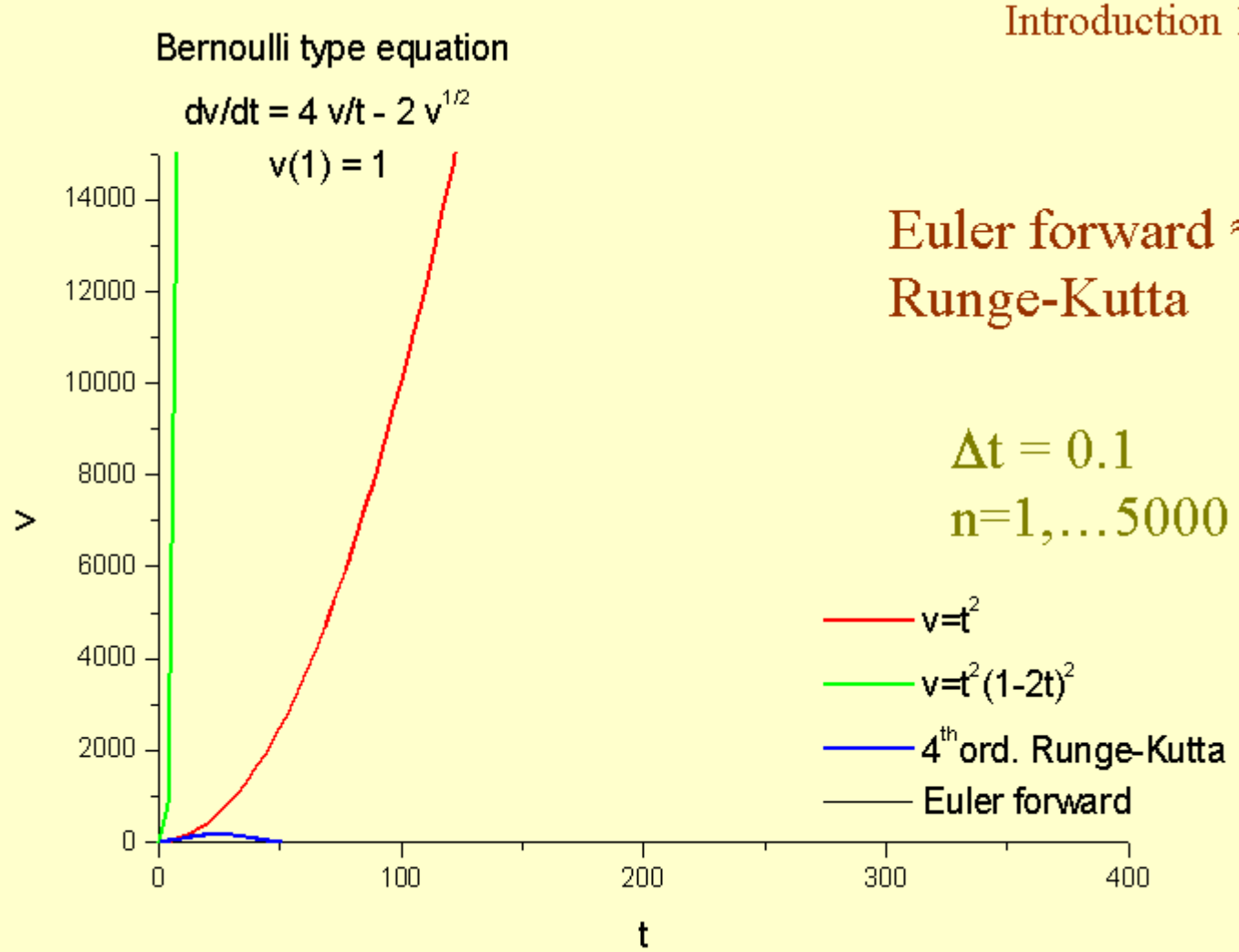
$n=1,\ldots 500000$



Euler forward ≈ Runge-Kutta

$\Delta t = 0.01$

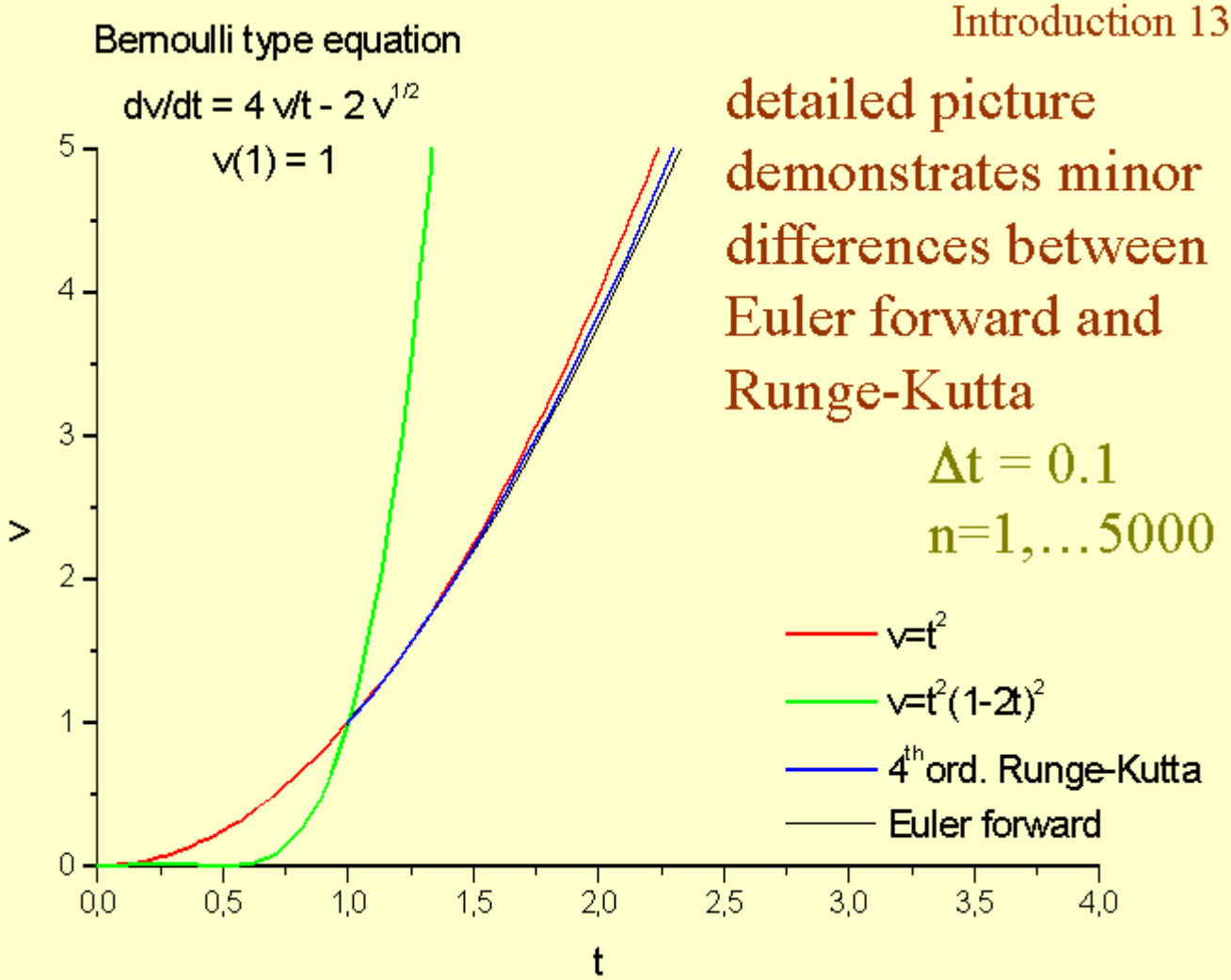
$n=1,\ldots 50000$



Euler forward ≈ Runge-Kutta

$\Delta t = 0.1$

$n=1,\ldots 5000$



detailed picture demonstrates minor differences between Euler forward and Runge-Kutta

$\Delta t = 0.1$

$n=1,\ldots 5000$

- Even very “sophisticated” schemes, like 4th order Runge-Kutta, may fail to reproduce the true solution, if the scheme is not properly designed!

- Numerical method my not reproduce all solutions.

Lewis Fry Richardson

1922 (before the discovery of CFL condition!):

Weather Prediction by Numerical Process

- complete set of equations
- dependant variables integrated over “strata”
- diagnostic equation for w!

F. Nebeker: *Calculating the Weather.* (1995, Acad. Press)
P. Lynch: *Richardson’s marvelous forecast.* In: M. Shapiro and S. Grønås: *Tle life cycle of extratropical cyclones.* p. 61 (1999, AMS)

Historical 2

International balloon day 20 May 1910

$\Delta\lambda=3°$

$\Delta\varphi=1.8°$

$z_0=h(\lambda,\varphi)$

$z_1=2.0$ km

$z_2=4.2$ km

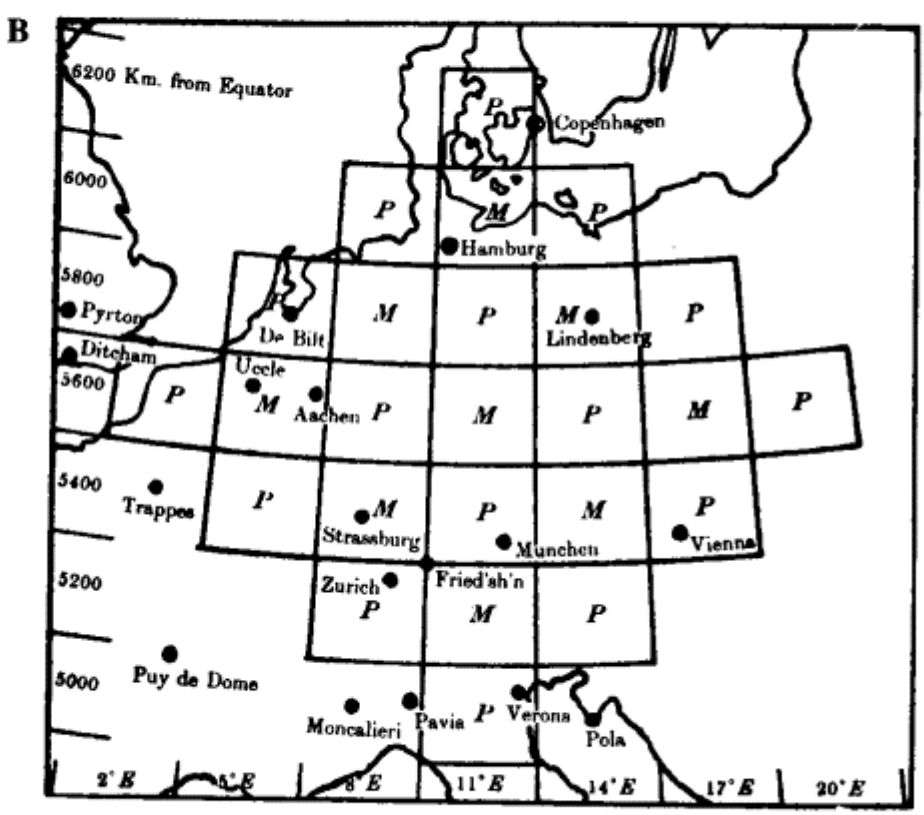
$z_3=7.2$ km

$z_4=11.8$ km

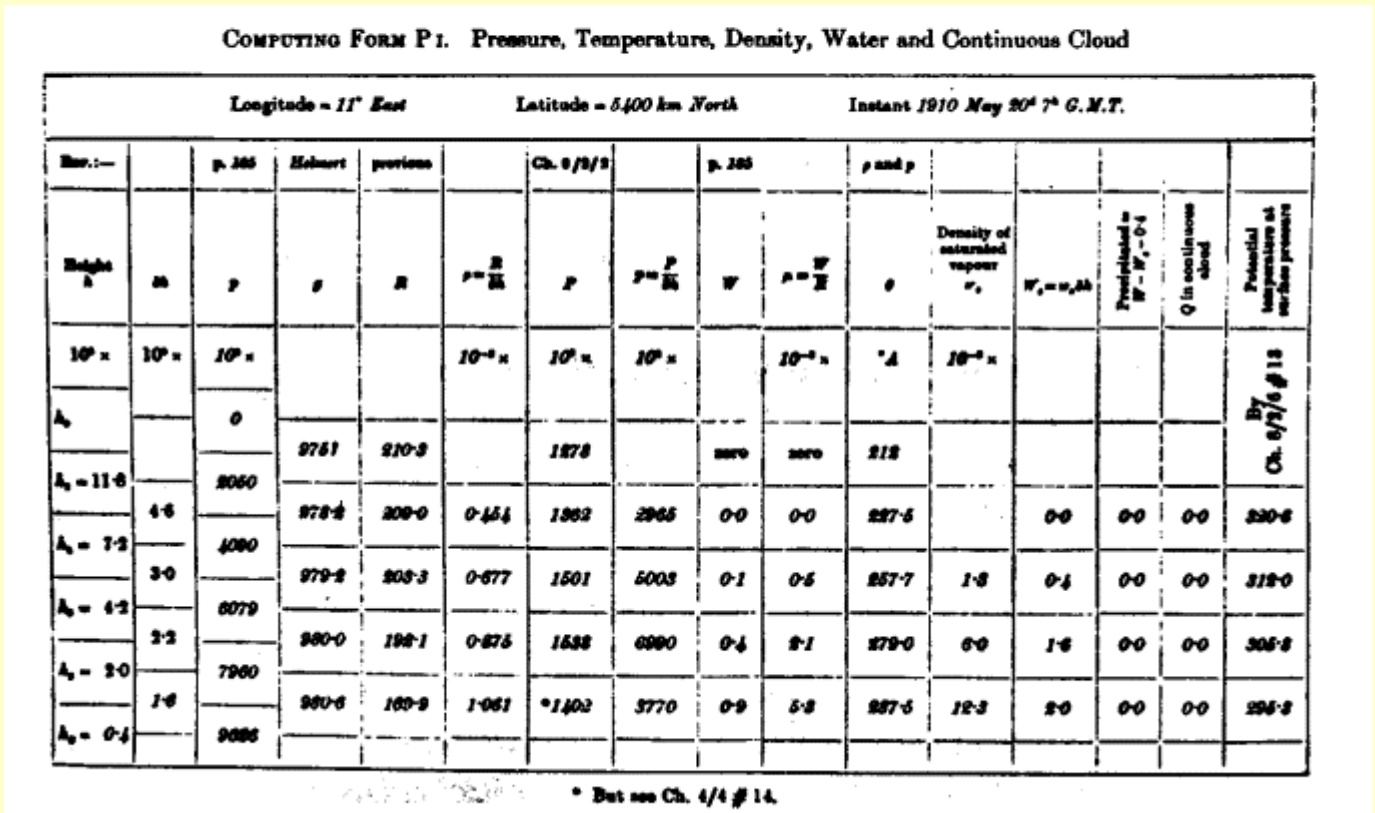
staggered E-grid, leap-frog scheme

P,W - points

R,U,V - points



One of the Richardson's 23 "computing forms"



COMPUTING FORM P I. Pressure, Temperature, Density, Water and Continuous Cloud

Longitude = 11° East Latitude = 5400 km North Instant 1910 May 20d 7h G.M.T.

Ref.:—		p. 185	Helmert	previous		Ch. 8/2/2		p. 185		ρ and p					
Height h	δh	p	g	R	$\rho = \frac{R}{\delta h}$	P	$p = \frac{P}{\delta h}$	W	$\mu = \frac{W}{R}$	θ	Density of saturated vapour w_s	$W_s = w_s \delta h$	Precipitated $= W - W_s - 0{\cdot}4$	Q in continuous cloud	Potential temperature at surface pressure
$10^3 \times$	$10^3 \times$	$10^3 \times$			$10^{-3} \times$	$10^3 \times$	$10^3 \times$		$10^{-3} \times$	°A	$10^{-3} \times$				By Ch. 8/2/6 # 13
h_0		0													
			9751	210·3		1278		zero	zero	212					
$h_1 = 11{\cdot}8$		2050													
	4·6		978·2	209·0	0·454	1362	2965	0·0	0·0	227·5		0·0	0·0	0·0	320·6
$h_2 = 7{\cdot}2$		4090													
	3·0		979·2	203·3	0·677	1501	5003	0·1	0·5	257·7	1·3	0·4	0·0	0·0	312·0
$h_3 = 4{\cdot}2$		6079													
	2·2		980·0	192·1	0·875	1538	6990	0·4	2·1	279·0	6·0	1·6	0·0	0·0	305·8
$h_4 = 2{\cdot}0$		7960													
	1·6		980·6	169·9	1·061	*1402	3770	0·9	5·3	287·5	12·3	2·0	0·0	0·0	295·3
$h_5 = 0{\cdot}4$		9686													

* But see Ch. 4/4 # 14.

Richardson:

$\Delta t = 3h$

$\Delta x = 200$ km

$\Delta x/\Delta t = c \approx 20$ m/s

CFL:

$\Delta t < \Delta x/c$

or

$\Delta x/\Delta t > \text{const}$

Nowadays, CFL:

$\Delta t \sim 10$ min

$\Delta x \sim 50$ km

$\Delta x/\Delta t \sim 80$ m/s

Richardson's "fail":

He computed a tendency of
145 hPa/6 hours in surface pressure
– being completely wrong!

But: Richardson was bold enough to publish this result!

P. Lynch (1999) repeated the forecast from the same initial conditions with a modern model (adapted to resemble the Richardson's one).

He computed the same tendency of
145 hPa/6 hours in surface pressure!

L.F. Richardson's 1922 and Peter Lynch's 1999 forecast of pressure tendency (hPa/6h)

level	Richardson	Lynch	Lynch digital filter
1	48.3	48.5	-0.2
2	77.0	76.7	-2.6
3	103.2	102.1	-3.0
4	126.5	124.5	-3.1
surface	145.1	145.4	-0.9

The reason of Richardson's "fail":

As with the same initial data one gets the same result also with a modern model:

- mainly due to non-filtered gravity waves from inconsistent initial fields

-only partly due to CFL criterion

Thus: from such inital fields the Richardson's forecast WAS RIGHT!!

Numerical schemes must ensure:

- consistency with analytical form of equation
- appropriate accuracy
- convergence to analytical solution
- stability of numerical solution

The concepts of

- consistency
- accuracy
- convergence
- stability

on a very simple example of
linear advective equation with finite differences.

F. Mesinger and A. Arakawa: *Numerical Methods used in Atmospheric Models*
(1976, WMO)
K. W. Morton and D. F. Mayers: *Numerical solution of partial differential equations*
(1994, Cambridge Univ. Press)

Advective equation

$$\left[\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0 \quad \text{non} - \text{linear}\right] \quad - \text{later!}$$

$$\frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x} = 0 \quad \text{linear}$$

$$\text{solution}: \quad v = f(x - ct)$$

$$x - ct = \text{const}: \quad \text{the characteristics}$$

Differential vs. finite-difference equation

$$\frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x} = 0 \Leftrightarrow \frac{\Delta v}{\Delta t} + c\frac{\Delta v}{\Delta x} = 0$$

A "scheme": a method of expressing the differential equation with finite-differences approximation. How?

Many, many possibilities!
The most “natural, direct” one:
forward in time, backward in space

$$\frac{V_i^{n+1} - V_i^n}{\Delta t} + c\,\frac{V_i^n - V_{i-1}^n}{\Delta x} = 0$$

$$V_i^{n+1} = V_i^n - c\Delta t\,\frac{V_i^n - V_{i-1}^n}{\Delta x}$$

Consistency of finite differences scheme with differential equation means:

$$\frac{V_i^{n+1} - V_i^n}{\Delta t} + c\frac{V_i^n - V_{i-1}^n}{\Delta x} = 0 \underset{\Delta t \to 0, \Delta x \to 0}{\Longrightarrow} \frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x} = 0$$

Is it so in our case?

$$V_i^{n+1} = V_i^n - c\Delta t \frac{V_i^n - V_{i-1}^n}{\Delta x}$$

Consistency?

$$V_i^{n+1} = V_i^n - c\Delta t \frac{V_i^n - V_{i-1}^n}{\Delta x}$$

Truncation error T(x,t) : a difference between the both sides of equation should vanish if the scheme is consistent.

$$T(x,t) = \frac{1}{\Delta t}\left[V(x,t+\Delta t) - V(x,t)\right]$$

$$+\frac{c}{\Delta x}\left[V(x+\Delta x,t) - V(x,t)\right] = 0$$

$$(\Delta t \to 0, \Delta x \to 0) \Rightarrow T(x,t) \to 0!$$ ☺!

Is it possible to construct a non-consistent scheme?

Almost NOT – if the equation is evident and if the modeller is skilled enough.

But: YES – if the equation is very complicated and if the modeller….

Accuracy of the scheme: the order of the residual of Taylor series

$$v(t+\Delta t) = v(t) + \frac{\partial v}{\partial t}(t)\Delta t + \frac{1}{2}\frac{\partial^2 v}{\partial t^2}(t)(\Delta t)^2 + \frac{1}{6}\frac{\partial^3 v}{\partial t^3}(t)(\Delta t)^3 + ...$$

(devide by Δt and re-arrange)

$$\frac{\partial v}{\partial t}(t) = \frac{v(t+\Delta t) - v(t)}{\Delta t} - \frac{1}{2}\frac{\partial^2 v}{\partial t^2}(t)(\Delta t) - \frac{1}{6}\frac{\partial^3 v}{\partial t^3}(t)(\Delta t)^2 - ....$$

$$= \frac{v(t+\Delta t) - v(t)}{\Delta t} - a(t)(\Delta t) - b(t)(\Delta t)^2 - ....$$

$$\frac{\partial v}{\partial t}(t) = \frac{v(t+\Delta t) - v(t)}{\Delta t} - O(\Delta t)$$

1st order accuracy in time

$$\frac{\partial v}{\partial t}(t) = \frac{v(t+\Delta t) - v(t)}{\Delta t} - O(\Delta t)$$

$$\Delta t \to 0 \quad \Rightarrow \quad \frac{v(t+\Delta t) - v(t)}{\Delta t} \to \frac{\partial v}{\partial t}$$

(If the scheme is at least of the 1st order it is also consistent ☺!)

2nd order in time

$$v(t+\Delta t) = v(t) + \frac{\partial v}{\partial t}(t)\Delta t + \frac{1}{2}\frac{\partial^2 v}{\partial t^2}(t)(\Delta t)^2 + \frac{1}{6}\frac{\partial^3 v}{\partial t^3}(t)(\Delta t)^3 + \ldots..$$

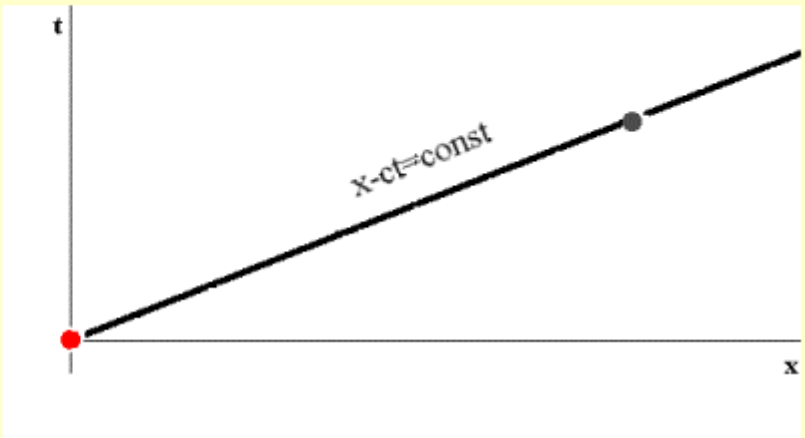
$$\frac{\partial v}{\partial t}(t) = \frac{v(t+\Delta t)-v(t)}{\Delta t} - \frac{1}{2}\frac{\dfrac{v(t+\Delta t)-v(t)}{\Delta t} - \dfrac{v(t)-v(t-\Delta t)}{\Delta t}}{\Delta t}\Delta t - \frac{1}{6}\frac{\partial^3 v}{\partial t^3}(t)(\Delta t)^2 - \ldots.$$

$$\frac{\partial v}{\partial t}(t) = \frac{v(t+\Delta t)-v(t)}{\Delta t} - \frac{1}{2}\frac{v(t+\Delta t)-2v(t)-v(t-\Delta t)}{\Delta t} - b(\Delta t)^2 - \ldots =$$

$$\frac{\partial v}{\partial t}(t) = \frac{v(t+\Delta t)-v(t-\Delta t)}{2\Delta t} - O(\Delta t^2)$$

Two time-steps scheme
is a 2nd order accuracy-in-time scheme!

Convergence towards the analytical solution



$$\frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x} = 0$$

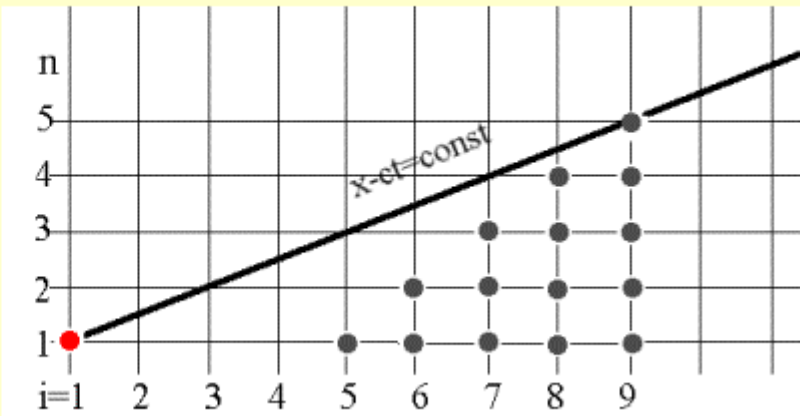
solution : $v = f(x - ct)$

$x - ct = \text{const} \Rightarrow v = \text{const}$

(along the characteristics)

Analytical solution in black point depends only on the value in red point!

Convergence

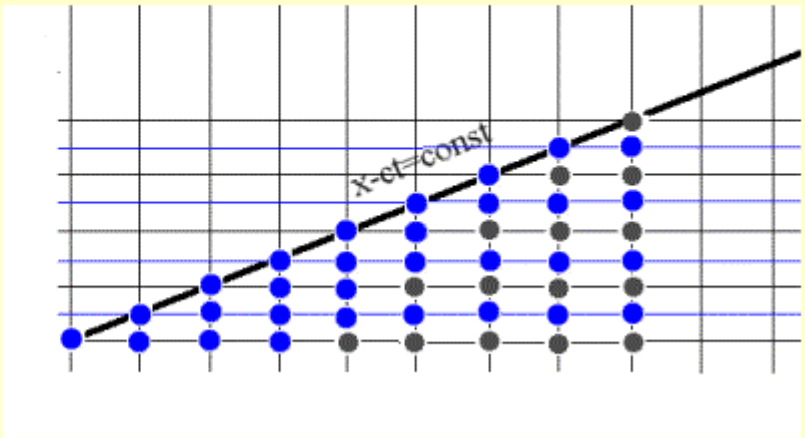


$$V_i^{n+1} = V_i^n - c\Delta t \frac{V_i^n - V_{i-1}^n}{\Delta x}$$

Numerically: the solution in next time step depends on two points in previous time step.
In this case no dependance of black point V(9,5) on red point in V(0,0) ☹!

Convergence

If the time resulution Δt is fine enough, the numerical solution solution in black point depends also on red point in V(0,0) ☺!



$$\frac{c\Delta t}{\Delta x} \leq 1$$

Condition of convergence

Stability of the scheme

$$\frac{V_i^{n+1} - V_i^n}{\Delta t} + c\frac{V_i^n - V_{i-1}^n}{\Delta x} = 0$$

$$V_i^{n+1} = V_i^n\left(1 - \frac{c\Delta t}{\Delta x}\right) + \frac{c\Delta t}{\Delta x}V_{i-1}^n$$

$$V_i^{n+1} = V_i^n(1-\mu) + \mu V_{i-1}^n$$

V should stay finite when

$t \to \infty$ ($n \to \infty$)!

Stability testing: the energy method

$$\sum \left(V_j^{n+1}\right)^2 \le A < \infty$$

$$\sum_{i=1}^{I} \left(V_i^{n+1}\right)^2 = \sum_{i=1}^{I} \left[(1-\mu)^2 \left(V_i^n\right)^2 + 2\mu(1-\mu) V_i^n V_{i-1}^n + \mu^2 \left(V_{i-1}^n\right)^2 \right]$$

cyclical boundary values $V_0 = V_I$, thus

$$\sum_{i=1}^{I} \left(V_{i-1}^n\right)^2 = \sum_{i=1}^{I} \left(V_i^n\right)^2$$

and (Schwarz relation)

$$\sum V_i^n V_{i-1}^n \le \sqrt{\sum \left(V_i^n\right)^2} \sqrt{\sum \left(V_{i-1}^n\right)^2} = \sum \left(V_i^n\right)^2$$

The sum of squares in next time step depends on the sum of squares in previous time step:

$$\sum_{i=1}^{I}\left(V_i^{n+1}\right)^2 \le \left[(1-\mu)^2 + 2\mu(1-\mu) + \mu^2\right]\sum_{i=1}^{I}\left(V_i^{n}\right)^2$$

$$\sum_{i=1}^{I}\left(V_i^{n+1}\right)^2 \le (1-\mu^2)\sum_{i=1}^{I}\left(V_i^{n}\right)^2$$

This is certainly true if , $1-\mu^2 \ge 0$ ☺!

$$\mu = \frac{c\Delta t}{\Delta x} \le 1$$

Stability of the scheme

V should stay finite when $t \to \infty$ ($n \to \infty$)!

$$\frac{c\Delta t}{\Delta x} \leq 1$$

Stability criterion: CFL condition

R. Courant, K.O. Freidrich, H. Lewy, 1928: Über die partiellen Differenzgelichungen der mathematischem Physik. *Math. Ann.* **100**, p. 32.

Other methods for testing stability:

direct method: $\left|V_j^{n+1}\right| \le A < \infty$

von Neumann's (spectral) method:

$$v(x,t) = \mathrm{Re}\left[\hat{V}(t)\, e^{ikt}\right]$$

$$V_j^n = \mathrm{Re}\left[\hat{V}^n\, e^{ikj\Delta x}\right]$$

Non-linearity

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0$$

$$v(x,t) = V(t)\sin kx$$

non-linearity produces in time new phenomena of new sizes!

$$v\frac{\partial v}{\partial x} = V^2(t)k\sin kx\cos kx$$

$$v\frac{\partial v}{\partial x} = \frac{V^2(t)k}{2}\sin 2kx$$

product : $k \Rightarrow 2k$

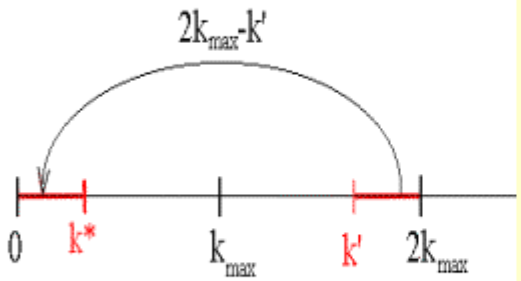
non-linearity and aliasing error

$$\sin k'x = \sin[\, 2k_{max}\, x - (2k_{max} - k')x\,] =$$

$$= \sin[\, 2\frac{\pi}{\Delta x} j\Delta x - -(2k_{max} - k')x\,] =$$

$$= -\sin(\, 2k_{max} - k')x = -\sin k^* x$$

$$\text{non} - \text{linearity} \Rightarrow k' \Rightarrow k^*$$



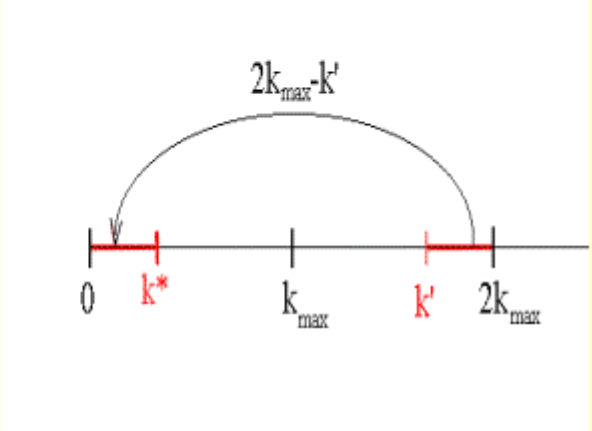
$$L_{min} = 2\Delta x \Rightarrow k_{max} = \frac{\pi}{\Delta x}$$

$$2k = k', \quad k' = 2k_{max} - (2k_{max}$$

$$x = j\Delta x$$

$$\text{non} - \text{linearity} \Rightarrow k' \Rightarrow k^*$$

non-linearity and aliasing error



Limited capability
of resolving the phenomena
causes that we “see” them falsely!

Why solving linear aproximations of non-linear equations analytically?

An example: conservation of vorticity

$$\frac{d}{dt}(\varsigma + f) = 0 \quad \text{non} - \text{linear}$$

$$\vec{v} = (u_0 + u', v', w') \Rightarrow \frac{d}{dt} = \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x}$$

$$\varsigma = \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}$$

$$u' = -\frac{\partial \Psi}{\partial y}, v' = \frac{\partial \Psi}{\partial x} \Rightarrow \varsigma = \nabla^2 \Psi$$

How much information can offer linearized equation being solved analytically?

$$\frac{d}{dt}(\varsigma+f)=0 \quad \Rightarrow \frac{\partial}{\partial t}\nabla^2\Psi+u_0\frac{\partial}{\partial x}\nabla^2\Psi+\beta\frac{\partial\Psi}{\partial x}=0$$

$$\Psi=\mathrm{Re}\left[\hat{\Psi}e^{i(kx+ly-\omega t)}\right]$$

$$\frac{\partial}{\partial x}\rightarrow ik, \quad \frac{\partial}{\partial t}\rightarrow -i\omega, \quad \nabla^2=-\left(k^2+l^2\right)$$

$$(-i\omega)\left[-\left(k^2+l^2\right)\right]\mathrm{Re}\left[\hat{\Psi}e^{i(kx+ly-\omega t)}\right]$$
$$+u_0 ik\left[-\left(k^2+l^2\right)\right]+\beta ik\,\mathrm{Re}\left[\hat{\Psi}e^{i(kx+ly-\omega t)}\right]=0$$

$$\omega = u_0 k - \frac{\beta k}{k^2+l^2}, \quad c_x = \frac{\omega}{k}$$

$$c_x = u_0 - \frac{\beta}{k^2+l^2}$$

Initially non-linear equation linearized: analytical information on Rossby waves

Analytical solution of a linearized case - a test of numerical model

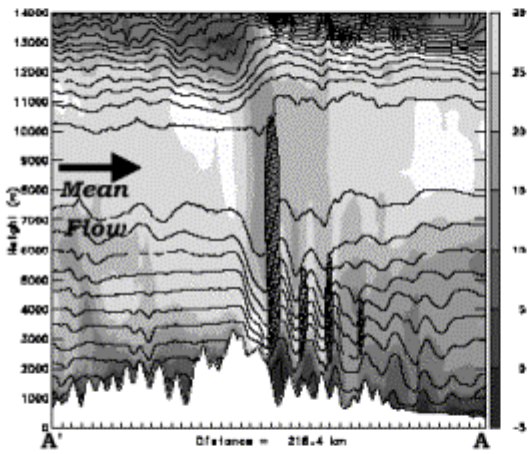


Fig. 3. COAMPS simulated cross-mountain (v) wind component (shaded, m s^{-1}) and potential temperature (every 3 K) for 1400 UTC 20 September (14 h). The vertical velocity in excess of 4 m s^{-1} is hatched.

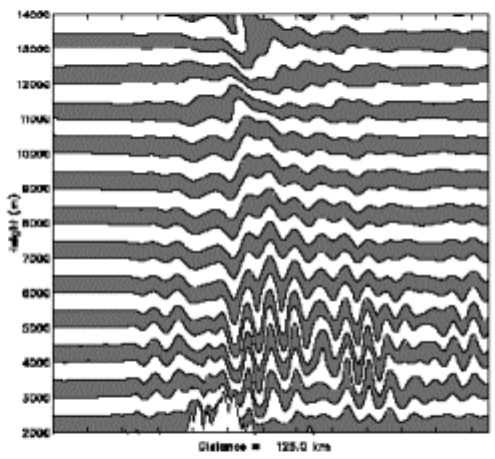


Fig. 4. Vertical displacement field from nonhydrostatic linear theory.

J. Doyle, R. Smith, G. Poulos (2001): Lee waves over complex topography. *MAP Newsletter* **15**, p. 40

Further comparisona with measurements - the final, decisive test!

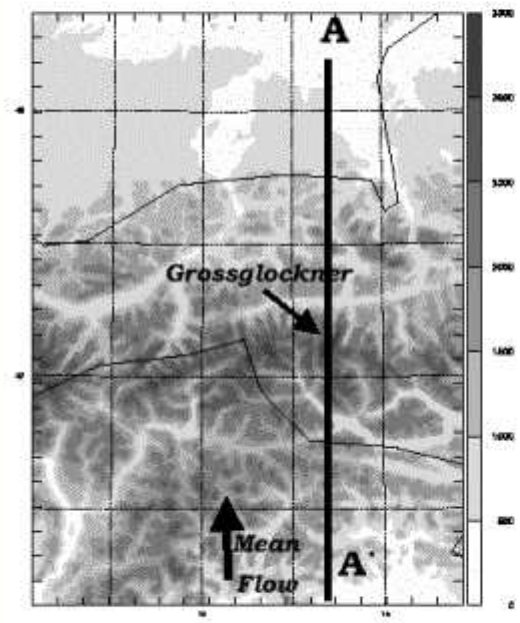


Fig. 1. The Electra flight track (AA') and topography (m) of the Hohe Tauern region

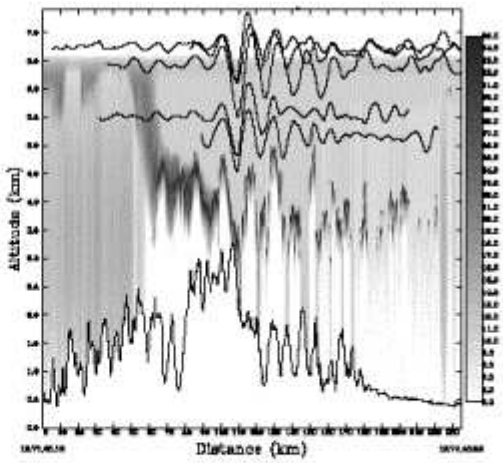


Fig. 2. The downlooking SABL backscatter coefficient and vertical parcel displacements from the Electra