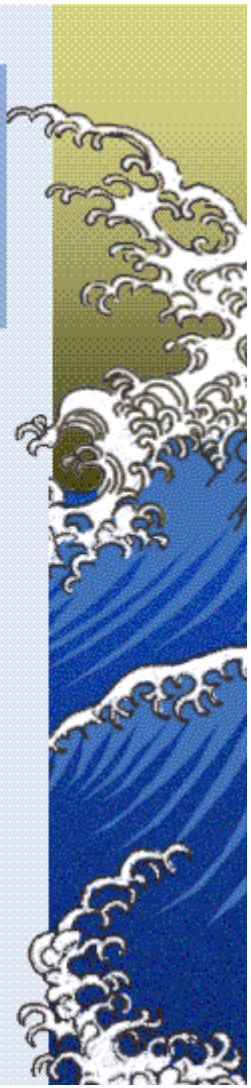
The observations in the variational cost function
Plan of the lecture

- General position of the problem
- The variational formulation
- The observation operators
- Horizontal and vertical interpolations
- Adjoint of the obs operators



The observations in the variational cost function

Let **x** be the control variable of the minimisation problem

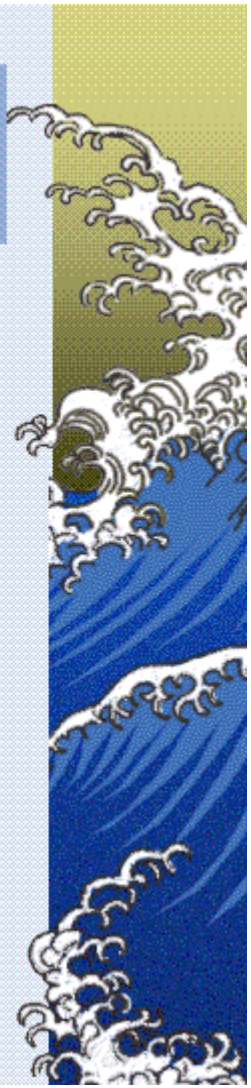
The available observations for a given time period are gathered into an observation vector **y**

To use it in an analysis, it is necessary to compare the observations with the state vector **x**

This is done through the use of a function from model state space to observation space called an observation operator, denoted by **H** generating values **H(x)**

H is a collection of interpolation operators from the model discretisation to the observation points, and conversions from model variables to the observed parameters

We call **y-H(x)** the vector of departures at the observation points. When calculated with background $\mathbf{x}_b$ it is called innovations, and with the analysis $\mathbf{x}_a$, analysis residuals



The observations in the variational cost function

General hypotheses and notations :

$\mathbf{X}_b$ background model state

$\mathbf{X}_a$ analysis model state

$\mathbf{y}$ vector of observations

$\mathbf{H}$ observation operator

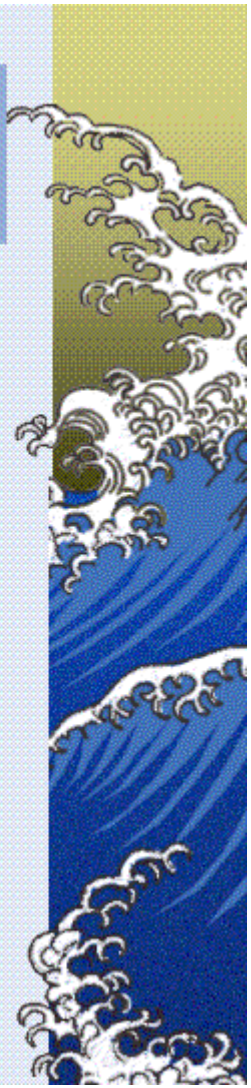
$\mathbf{B}$ covariance matrix of the background errors

$\mathbf{R}$ covariance matrix of observation errors

$\mathbf{B}$ and $\mathbf{R}$ positive definite matrices

Unbiased and uncorrelated errors

Optimal analysis : we look for an analysis state which is as close as possible to the true state in a rms sens $\Rightarrow$ least-squares analysis equations



The observations in the variational cost function

The optimal least-square estimator (or BLUE analysis) is defined by :

$$x_a = x_b + K(y - H[x_b])$$

$$K = BH^T(HBH^T+R)^{-1}$$

Where the linear operator K is called gain, or weight matrix, of the analysis

The BLUE analysis is equivalently obtained as a solution to the variational problem :

$$X_a = \min J$$

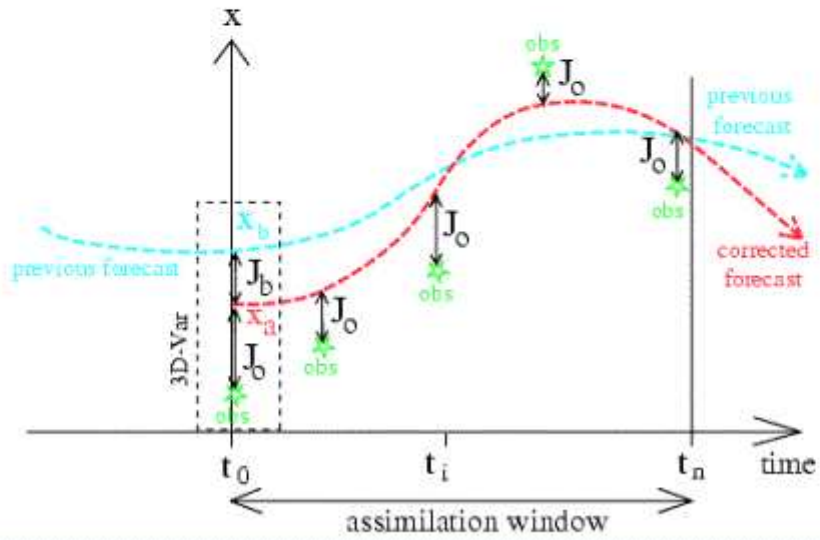
$$J(x) = (x-x_b)^T B^{-1}(x-x_b) + (y - H[x_b])^T R^{-1}(y - H[x_b])$$

$$= J_b(x) + J_o(x)$$

Where J is called the cost function of the analysis, J_b is the background term and J_o is the observation term

The observations in the variational cost function

Compared properties of 3D-Var and 4D-Var



The observations in the variational cost function

$$J_o(x) = (y - H[x_b])^T R^{-1} (y - H[x_b])$$

The sets of observations of different types have errors that are uncorrelated

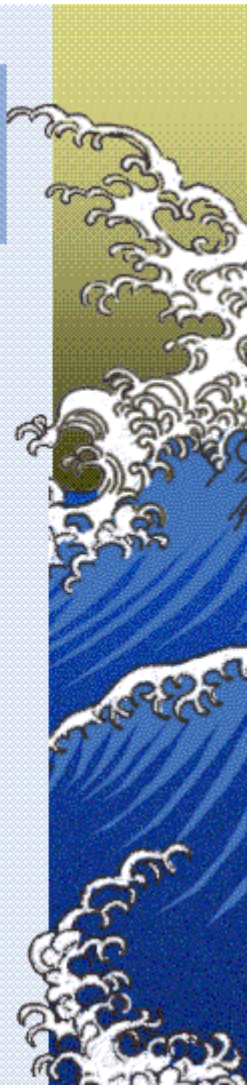
$\Rightarrow$ R is a block-diagonal matrix. It gives :

$$J_o(x) = \sum (y_i - H_i[x_b])^T R_i^{-1} (y_i - H_i[x_b])$$

Where **i** denotes the types of observation, and R_i the obs error covariances matrices for each type

The thinning of the observations yields to the assumption that the observations are independent (not correlated in the horizontal)

$\Rightarrow$ The matrices R_i are also block-diagonal, each little block corresponding to the error covariances between the data of one observation (for example between the geopotential data of a radiosonde or between the radiances of a satellite observation)



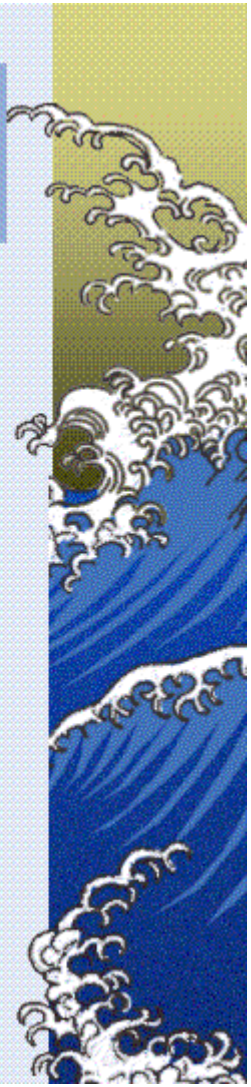
The observations operators in the variational cost function

We have seen that the J_o part of the cost function can be finally written : $J_o(x) = \sum \sum (y_{ik} - H_{ik}[x_b])^T R_{ik}^{-1} (y_i - H_{ik}[x_b])$

Where H_{ik} is the observation operator for obs n° k of type n° i.

The opérator H_{ik} is subdivided into a sequence of operators, each one of which performs part of the transformation from control variable to observed parameters :

- Conversion from control variable to model variables
- Inverse spectral transforms put the model variables on the gridpoint space
- A horizontal interpolation provides vertical profiles of model variables at observation locations
- Vertical integration if necessary (hydrostatic equation for geopotential, radiative transfer equation for radiances...)
- Vertical interpolation to the level of the observations



The observations operators in the variational cost function

Horizontal interpolations :

• A 12-point bi-cubic or 4-point bi-linear horizontal interpolation gives vertical profiles of model variables at observation locations. The surface fields are interpolated bilinearly to avoid spurious maxima and minima

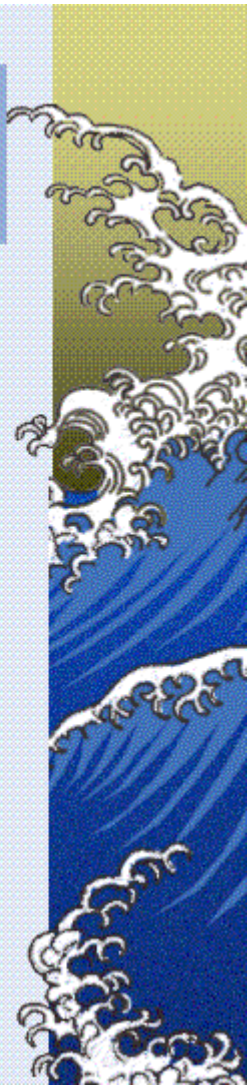
Vertical interpolations : it depends on the variable

• Linear in pressure for temperature and specific humidity

• Linear in logarithm of pressure for wind

• Linear in logarithm of pressure for geopotential (performed in term of departures from the ICAO standard atmosphere). New operator until a few years, specific to Météo-France, better consistent with hydrostatism.

• Vertical interpolations for surface observations (T2m, V10m, Hu2m) are done consistently with the physics of the model



The observations in the variational cost function

Adjoints of the observation operators

The solution of the minimisation problem is given by $\nabla J(x_a) = 0$

At each step of the descent algorithm, ∇J has to be computed :

$$\nabla J(x) = 2B^{-1}(x - x_b) - 2H^T R^{-1}(y - H[x_b])$$

Where H^T is the the adjoint ($\Leftrightarrow$transpose) of the the observation operator (or the tangent linear of the observation operator when it is non-linear).

It means that the adjoints of the obs operators have to be computed, which is easy for the interpolations, but more difficult for the vertical integrations, in particular for the radiative transfer equation or other operators containing non differentiable processes.