Physical Oceanography - UNAM, Mexico
Lecture 2: The Equations of Ocean Circulation and Ocean Modelling

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A first taste...

$$
\begin{aligned}
\frac{\partial \overline{\mathbf{u_h}}}{\partial t} + (\overline{\mathbf{u}}.\boldsymbol{\nabla})\overline{\mathbf{u_h}} + f\mathbf{k}\times\overline{\mathbf{u_h}} &= -\frac{1}{\rho_0}\boldsymbol{\nabla_h}\overline{P} + \boldsymbol{\nabla_h}.(\kappa_{hu}\boldsymbol{\nabla_h})\overline{\mathbf{u_h}} + \frac{\partial}{\partial z}(\kappa_{zu}\frac{\partial \overline{\mathbf{u_h}}}{\partial z}) \\
\overline{P}(z) &= \rho_0 g\overline{\eta} + g\int_z^0 \overline{\rho}\,dz' \\
\overline{w}(z) &= -\int_{-H}^{z} \boldsymbol{\nabla_h}.\overline{\mathbf{u_h}}\,dz' \\
\frac{\partial \overline{\eta}}{\partial t} &= -\boldsymbol{\nabla_h}.\int_{-H}^{\eta}\overline{\mathbf{u_h}}\,dz + \overline{P} + \overline{R} - \overline{E} \\
\frac{\partial \overline{\theta}}{\partial t} + (\overline{\mathbf{u}}.\boldsymbol{\nabla})\overline{\theta} &= \boldsymbol{\nabla_h}.(\kappa_{hT}\boldsymbol{\nabla_h})\overline{\theta} + \frac{\partial}{\partial z}(\kappa_{zT}\frac{\partial \overline{\theta}}{\partial z}) + \frac{1}{\rho c_w}\overline{\dot{\Theta}} \\
\frac{\partial \overline{S}}{\partial t} + (\overline{\mathbf{u}}.\boldsymbol{\nabla})\overline{S} &= \boldsymbol{\nabla_h}.(\kappa_{hS}\boldsymbol{\nabla_h})\overline{S} + \frac{\partial}{\partial z}(\kappa_{zS}\frac{\partial \overline{S}}{\partial z}) + \overline{\dot{S}} \\
\overline{\rho} &= \overline{\rho}(\overline{\theta}, \overline{S}, \overline{P_0}(z))
\end{aligned}
$$

All the physics of an ocean circulation model is here !

Outline

Outline

Conservation of mass : continuity

In the following : control volume of zonal, meridional and vertical sizes δx, δy and δz and density ρ on a fixed Cartesian coordinate system $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ attached to the ground.

Conservation of mass : continuity

Mass conservation :

Conservation of mass : continuity

Mass conservation :

$$\begin{aligned}
\frac{d(\rho\delta x\delta y\delta z)}{dt} &= 0 \\
&= \delta x\delta y\delta z\frac{d\rho}{dt} + \rho(\delta y\delta z\frac{d\delta x}{dt} + \delta x\delta z\frac{d\delta y}{dt} + \delta x\delta y\frac{d\delta z}{dt}) \\
&= \delta x\delta y\delta z\frac{d\rho}{dt} + \rho(\delta y\delta z\delta u + \delta x\delta z\delta v + \delta x\delta y\delta w)
\end{aligned}$$

Conservation of mass : continuity

Mass conservation :

$$
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\frac{d(\rho\delta x\delta y\delta z)}{dt} &= 0 \\
&= \delta x\delta y\delta z\frac{d\rho}{dt}+\rho(\delta y\delta z\frac{d\delta x}{dt}+\delta x\delta z\frac{d\delta y}{dt}+\delta x\delta y\frac{d\delta z}{dt}) \\
&= \delta x\delta y\delta z\frac{d\rho}{dt}+\rho(\delta y\delta z\delta u+\delta x\delta z\delta v+\delta x\delta y\delta w)
\end{aligned}
$$

Hence dividing by $\delta x\delta y\delta z$:

$$
\begin{aligned}
\frac{d\rho}{dt}+\rho(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}) &= 0 \\
\iff \frac{d\rho}{dt}+\rho\boldsymbol{\nabla}.\mathbf{v} &= 0
\end{aligned}
$$

with $\boldsymbol{\nabla}=(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})$ the space derivative operator.

Conservation of mass : continuity

In the ocean, Boussinesq approximation : relative density variations are small :
$\rho(x,y,z,t) = \rho_0 + \rho'(x,y,z,t)$ and $\rho' << \rho_0$

The continuity equation becomes :

Conservation of mass : continuity

In the ocean, Boussinesq approximation : relative density variations are small :
$\rho(x,y,z,t) = \rho_0 + \rho'(x,y,z,t)$ and $\rho' << \rho_0$

The continuity equation becomes :
$\frac{d\rho'}{dt} + (\rho_0 + \rho')\boldsymbol{\nabla}.\mathbf{v} \simeq \rho_0 \boldsymbol{\nabla}.\mathbf{v} = 0$

$\iff \boldsymbol{\nabla}.\mathbf{v} = 0$

- To a very good approximation, oceanic currents are non-divergent.

Conservation of momentum

Newton's 2nd law : the Lagrangian (material) evolution of momentum is determined by the sum of external (gravity) and body (pressure and friction) forces.

Conservation of momentum

The acceleration in the Eulerian (fixed) framework $\frac{\partial \mathbf{u}}{\partial t}$ is far easier to observe and model than the Lagrangian one $\frac{d\mathbf{u}}{dt}$.
Relation between both :

Conservation of momentum

The acceleration in the Eulerian (fixed) framework $\frac{\partial \mathbf{u}}{\partial t}$ is far easier to observe and model than the Lagrangian one $\frac{d\mathbf{u}}{dt}$.
Relation between both :

$$\begin{aligned}
\delta \mathbf{u} &= \frac{\partial \mathbf{u}}{\partial t}\delta t + \frac{\partial \mathbf{u}}{\partial x}\delta x + \frac{\partial \mathbf{u}}{\partial y}\delta y + \frac{\partial \mathbf{u}}{\partial z}\delta z \\
\Longleftrightarrow \frac{d\mathbf{u}}{dt} &= \frac{\partial \mathbf{u}}{\partial t}\frac{dt}{dt} + \frac{\partial \mathbf{u}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{u}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{u}}{\partial z}\frac{dz}{dt} \\
&= \frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{u}}{\partial x}u + \frac{\partial \mathbf{u}}{\partial y}v + \frac{\partial \mathbf{u}}{\partial z}w \\
&= \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}.\boldsymbol{\nabla})\mathbf{u}
\end{aligned}$$

$(\mathbf{u}.\boldsymbol{\nabla})\mathbf{u}$ is the (nonlinear) advection term.

Conservation of momentum

Advection of zonal momentum in all faces of the control volume :

Conservation of momentum

Advection of zonal momentum in all faces of the control volume :

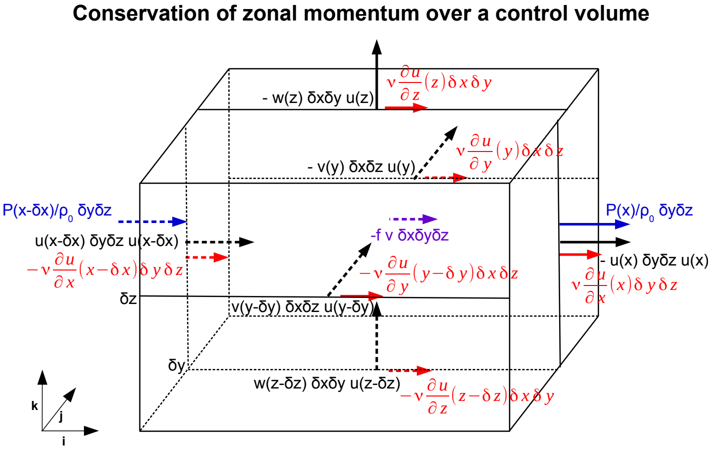

Figure 1 – Conservation of zonal momentum over a control volume : advection (black), viscous forces (red), pressure forces (blue) and Coriolis acceleration (purple).

Conservation of momentum

Zonal pressure force on the western and eastern faces of the control volume :

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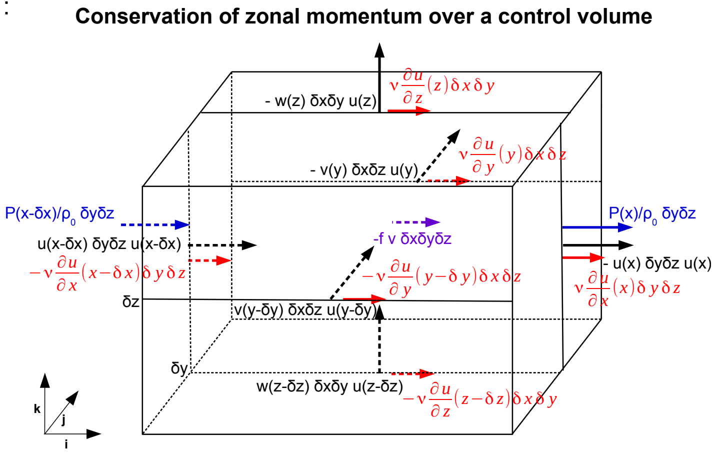

Figure 2 – Conservation of zonal momentum over a control volume : advection (black), viscous forces (red), pressure forces (blue) and Coriolis acceleration (purple).

Conservation of momentum

Zonal pressure force on the western and eastern faces of the control volume :

$$F_{Px} = (P(x - \delta x) - P(x))\delta y \delta z$$

hence volumic zonal pressure force :

$$\frac{F_{Px}}{\delta x \delta y \delta z} = -\frac{\partial P}{\partial x}$$

Generalizing to other spatial dimensions :

$$\frac{\mathbf{F_P}}{\delta x \delta y \delta z} = -\boldsymbol{\nabla} P$$

Conservation of momentum

Friction $\boldsymbol{F}_\tau$:

- Newton's law of viscosity for a Boussinesq Newtonian fluid :

$$\tau_{ij} = +\nu\rho(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j})$$

  with τ_{ij} the viscous stress exerted over the coordinate i on velocity component j, $x_i, x_j = (x, y, z)$, $u_i, u_j = (u, v, w)$ and $\nu = 8.9 \times 10^{-7} m^2/s$ the water kinematic viscosity.
- The second term vanishes in a Boussinesq fluid : hence friction behaves just like molecular or heat diffusion !

Conservation of momentum

Zonal friction force on all faces of the control volume :

Conservation of momentum

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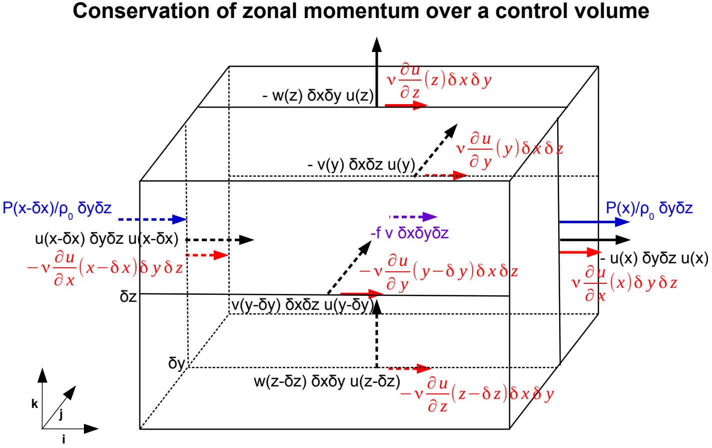

Figure 3 – Conservation of zonal momentum over a control volume : advection (black), viscous forces (red), pressure forces (blue) and Coriolis acceleration (purple).

Conservation of momentum

Zonal friction force on all faces of the control volume :

$$
\begin{aligned}
F_{\tau x} &= (-\tau_{xx}(x-\delta x)+\tau_{xx}(x))\delta y \delta z + (-\tau_{yx}(y-\delta y)+\tau_{yx}(y))\delta x \delta z \\
&+ (-\tau_{zx}(z-\delta z)+\tau_{zx}(z))\delta x \delta y \\
&= -\nu\rho\left[\frac{\partial}{\partial x}(x-\delta x)\delta y\delta z + \frac{\partial}{\partial y}(y-\delta y)\delta x\delta z + \frac{\partial}{\partial z}(z-\delta z)\delta x\delta y\right. \\
&- \left.\frac{\partial}{\partial x}(x)\delta y\delta z + \frac{\partial}{\partial y}(y)\delta x\delta z + \frac{\partial}{\partial z}(z)\delta x\delta y\right] u \\
&= +\nu\rho\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u\delta x\delta y\delta z \\
&= +\nu\rho\Delta u\delta x\delta y\delta z
\end{aligned}
$$

with $\Delta = \boldsymbol{\nabla}^2$ the Laplacian operator. Hence on the volume control over the three dimensions, the volumic friction force is :

$$
\frac{\mathbf{F}_\tau}{\delta x\delta y\delta z} = +\nu\rho\Delta\mathbf{u}
$$

Conservation of momentum

Adding the gravity force, Newton's 2nd law writes as :

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\begin{aligned}
\rho\delta x\delta y\delta z(\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}.\boldsymbol{\nabla})\mathbf{u}) &= \delta x\delta y\delta z(-\boldsymbol{\nabla}P+\nu\rho\Delta u-\rho g\mathbf{k}) \\
\Longleftrightarrow \frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}.\boldsymbol{\nabla})\mathbf{u} &= -\frac{1}{\rho}\boldsymbol{\nabla}P+\nu\Delta\mathbf{u}-g\mathbf{k}
\end{aligned}
$$

with g the gravity acceleration and $\mathbf{k}$ the vertical unit vector.

Conservation of momentum

Hydrostatic assumption for the vertical momentum equation :

$$\frac{\partial P}{\partial z} = -\rho g$$

Conservation of momentum

Second Boussinesq approximation for the horizontal momentum equations : $P = P_0 + P'$ with $P' << P_0$ as a consequence of $\rho' << \rho_0$.

Volumic momentum equations with respect to a reference state at rest with $\frac{\partial P_0}{\partial z} = -\rho_0 g$:

Conservation of momentum

Second Boussinesq approximation for the horizontal momentum equations : $P = P_0 + P'$ with $P' << P_0$ as a consequence of $\rho' << \rho_0$.

Volumic momentum equations with respect to a reference state at rest with $\frac{\partial P_0}{\partial z} = -\rho_0 g$:

$$
\begin{aligned}
(\rho_0+\rho')(\frac{\partial \mathbf{u_h}'}{\partial t} + (\mathbf{u}'.\nabla)\mathbf{u_h}') &= -\nabla P' + \nu(\rho_0+\rho')\Delta \mathbf{u_h}' - \rho' g \mathbf{k} \\
\Longrightarrow \rho_0(\frac{\partial \mathbf{u_h}'}{\partial t} + (\mathbf{u}'.\nabla)\mathbf{u_h}') &= -\nabla P' + \nu\rho_0\Delta \mathbf{u_h}' - \rho' g \mathbf{k} \\
\Longrightarrow \frac{\partial \mathbf{u_h}'}{\partial t} + (\mathbf{u}'.\nabla)\mathbf{u_h}' &= -\frac{1}{\rho_0}\nabla P' + \nu\Delta \mathbf{u_h}' - \frac{\rho'}{\rho_0} g \mathbf{k}
\end{aligned}
$$

$b = -\frac{\rho'}{\rho_0} g$ is the buoyancy acceleration. It is the only means by which gravity impacts the dynamics.

Conservation of momentum

Second Boussinesq approximation for the horizontal momentum equations : $P = P_0 + P'$ with $P' << P_0$ as a consequence of $\rho' << \rho_0$.

Volumic momentum equations with respect to a reference state at rest with $\frac{\partial P_0}{\partial z} = -\rho_0 g$:

$$\begin{aligned}
(\rho_0+\rho')(\frac{\partial \mathbf{u_h'}}{\partial t} + (\mathbf{u}'.\nabla)\mathbf{u_h'}) &= -\nabla P' + \nu(\rho_0+\rho')\Delta\mathbf{u_h'} - \rho' g\mathbf{k} \\
\Longrightarrow \rho_0(\frac{\partial \mathbf{u_h'}}{\partial t} + (\mathbf{u}'.\nabla)\mathbf{u_h'}) &= -\nabla P' + \nu\rho_0\Delta\mathbf{u_h'} - \rho' g\mathbf{k} \\
\Longrightarrow \frac{\partial \mathbf{u_h'}}{\partial t} + (\mathbf{u}'.\nabla)\mathbf{u_h'} &= -\frac{1}{\rho_0}\nabla P' + \nu\Delta\mathbf{u_h'} - \frac{\rho'}{\rho_0} g\mathbf{k}
\end{aligned}$$

$b = -\frac{\rho'}{\rho_0} g$ is the buoyancy acceleration. It is the only means by which gravity impacts the dynamics.

The total momentum equation (reference plus perturbation) becomes :

$$\Longrightarrow \frac{\partial \mathbf{u_h}}{\partial t} + (\mathbf{u}.\nabla)\mathbf{u_h} = -\frac{1}{\rho_0}\nabla P + \nu\Delta\mathbf{u_h} - \frac{\rho'}{\rho_0} g\mathbf{k}$$

Conservation of momentum

Complications arising from the Earth's rotation and spherical shape :

- Earth's rotation : the local coordinate accelerates with respect to a global non-rotating coordinate system. The Coriolis and centrifugal accelerations arise.
- Spherical shape of the Earth's surface : the coordinate system rotates when parcels move. The metric terms arise.

Metric terms are not considered in the following : Cartesian coordinates.

Conservation of momentum

Material evolution of a parcel location $\mathbf{r}$ in the absolute frame related to the relative local coordinates :

Conservation of momentum

Material evolution of a parcel location **r** in the absolute frame related to the relative local coordinates :

$$
\begin{aligned}
(\frac{d\mathbf{r}}{dt})_A &= (\frac{d\mathbf{r}}{dt})_R + \boldsymbol{\Omega} \times \mathbf{r} \\
\Longleftrightarrow \ \mathbf{u_A} &= \mathbf{u_R} + \boldsymbol{\Omega} \times \mathbf{r}
\end{aligned}
$$

with Ω the Earth's angular velocity.

Conservation of momentum

A second derivation yields the correspondence of accelerations :

$$(\frac{d\mathbf{u_R}}{dt})_A = (\frac{d\mathbf{u_R}}{dt})_R + \mathbf{\Omega} \times \mathbf{u_R}$$

and with :

$$\begin{aligned}
(\frac{d\mathbf{u_R}}{dt})_A &= (\frac{d\mathbf{u_A}}{dt})_A - \frac{d}{dt}(\mathbf{\Omega} \times \mathbf{r})_A \\
&= (\frac{d\mathbf{u_A}}{dt})_A - \mathbf{\Omega} \times (\frac{d\mathbf{r}}{dt})_A \\
&= (\frac{d\mathbf{u_A}}{dt})_A - \mathbf{\Omega} \times (\mathbf{u_R} + \mathbf{\Omega} \times \mathbf{r})
\end{aligned}$$

we have :

Conservation of momentum

A second derivation yields the correspondence of accelerations :

$$(\frac{d\mathbf{u_R}}{dt})_A = (\frac{d\mathbf{u_R}}{dt})_R + \mathbf{\Omega} \times \mathbf{u_R}$$

and with :

$$\begin{aligned}
(\frac{d\mathbf{u_R}}{dt})_A &= (\frac{d\mathbf{u_A}}{dt})_A - \frac{d}{dt}(\mathbf{\Omega} \times \mathbf{r})_A \\
&= (\frac{d\mathbf{u_A}}{dt})_A - \mathbf{\Omega} \times (\frac{d\mathbf{r}}{dt})_A \\
&= (\frac{d\mathbf{u_A}}{dt})_A - \mathbf{\Omega} \times (\mathbf{u_R} + \mathbf{\Omega} \times \mathbf{r})
\end{aligned}$$

we have :

$$(\frac{d\mathbf{u_R}}{dt})_A = (\frac{d\mathbf{u_A}}{dt})_R + 2\mathbf{\Omega} \times \mathbf{u_R} + \mathbf{\Omega} \times \mathbf{\Omega} \times \mathbf{r}$$

Extra term 1 is Coriolis, extra term 2 is centrifugal acceleration.

Conservation of momentum

- Re-writing of the Coriolis acceleration : $2\mathbf{\Omega} \times \mathbf{u_R} \simeq (-fu_R, +fv_R, 0)$ with $f = 2\Omega sin(\phi)$ (with ϕ the latitude) the Coriolis parameter. We have neglected the horizontal Coriolis acceleration (with $w << u, v$ and the hydrostatic assumption).

Conservation of momentum

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- The centrifugal acceleration is mostly vertical and compensated for by the Earth's deformation at low latitudes. We hence include it in an effective gravity force :

$$\mathbf{g}^* = \mathbf{g} + \mathbf{\Omega} \times \mathbf{\Omega} \times \mathbf{r} \simeq -g^*\mathbf{k}$$

Conservation of momentum

Hence only horizontal Coriolis acceleration enters the momentum equations :

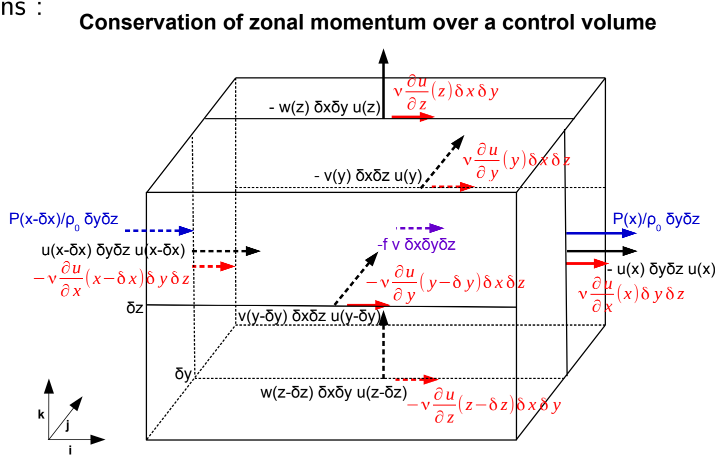

Figure 4 – Conservation of zonal momentum over a control volume : advection (black), viscous forces (red), pressure forces (blue) and Coriolis acceleration (purple).

Conservation of momentum

Final momentum equations under the hypotheses of Boussinesq, hydrostatism, tangent plane and the neglect of small terms related to the Earth's rotation :

$$\frac{\partial \mathbf{u_h}}{\partial t} + (\mathbf{u}.\boldsymbol{\nabla})\mathbf{u_h} + f\mathbf{k} \times \mathbf{u_h} = -\frac{1}{\rho_0}\boldsymbol{\nabla} P + \nu \Delta \mathbf{u_h} - g^* \mathbf{k}$$

Conservation of heat

Conservation of heat (first law of thermodynamics) over the control volume :

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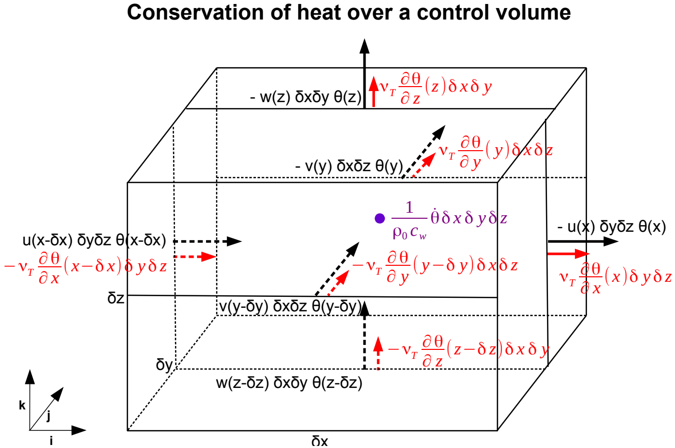

Figure 5 – Conservation of heat over a control volume : advection (black), diffusion (red), and source/sink (purple).

Conservation of heat

Conservation of temperature :

$$\frac{\partial \theta}{\partial t} + (\mathbf{u}.\boldsymbol{\nabla})\theta = \nu_T \Delta \theta + \frac{1}{\rho_0 c_w} \dot{\Theta}$$

with ν_T the thermal diffusivity of water, c_w the water heat capacity, $\dot{\Theta}$ (in W/m^3) sources and sinks of heat and θ the seawater potential temperature.

- Potential temperature : equivalent temperature if parcel uplifted adiabatically to surface. Limited compressibility of sea water, hence limited pressure correction.
- $\dot{\Theta}$ represents air-sea heat exchanges (and ice formation/fusion in the presence of sea ice).

Equation of state

- Unlike the atmosphere, no analytical equation relating density ρ to the other thermodynamic variables.
- Oceanographers use an empirical 78-member polynomial function of salinity, potential temperature and pressure to deduce seawater density.

Equation of state

- Density not linear in θ, S and P : in particular the thermal expansion varies between $\sim -0.05 kg/m^3/^\circ C$ for $\theta = 0^\circ C$ and $\sim -0.35 kg/m^3/^\circ C$ for $\theta = 30^\circ C$ (at $S = 35$ ‰ and $P = P_a$).

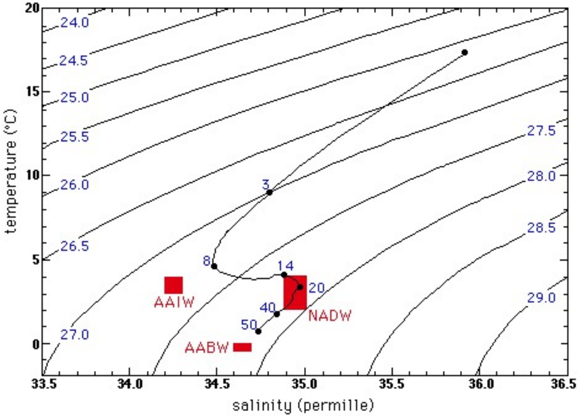

Figure 6 – (θ, S) diagram of a profile at $9^\circ S$ in the Atlantic (depth in hm, main water masses in red, BM's lecture).

Equation of state

A reasonable formula is given by the inclusion of two second-order terms accounting for the main nonlinearities of density : cabbeling and thermobaricity.

- Cabbeling is the systematic densification of seawater by mixing.
- Thermobaricity is the small dependency of thermal expansion on pressure.

The equation writes for the specific volume $v = \frac{1}{\rho}$:

$$v = v_0 \left[1 + \alpha_\theta (1 + \gamma^* P)(\theta - \theta_0) + \alpha_\theta^* (\theta - \theta_0)^2 - \beta_S (S - S_0) - \beta_P (P - P_0) \right]$$

with $(v_0, \theta_0, S_0, P_0)$ a reference state, $\beta = \frac{1}{\rho} \frac{\partial \rho}{\partial S}$ the haline contraction coefficient, $\beta_P = \frac{1}{\rho} \frac{\partial \rho}{\partial P}$ the compressibility coefficient, $\alpha_\theta^* = -\frac{1}{\rho} \frac{\partial^2 \rho}{\partial \theta^2}$ the second thermal expansion (or cabbeling) coefficient and $\gamma^* = \frac{\partial \alpha_\theta}{\partial P}$ the thermobaric parameter.

Equation of state

The nonlinearity of seawater with respect to θ and S has important consequences :

- Although density is conserved as θ and S are, its conservation equation is complicated by the involvement of nonlinear terms, so that it is usually not explicitely formulated.
- The ocean is more expanded from surface warming in the Tropics than contracted from surface cooling in the high latitudes, although the net heat flux is balanced. This average surface expansion of the global ocean, which would be equivalent to a heat imbalance of $Q_0 \sim +5W/m^2$, must be equilibrated otherwise the ocean would be ever expanding. It is indeed balanced by cabbeling which contracts the global ocean by mixing.

Equation of state

For most oceanic applications, thermobaricity can be ignored, so that a potential density referenced at surface is the most commonly used density variable :

$$\sigma_0 = \rho(S, \theta, P = P_a) - 1000$$

Conservation of salt

The equation of state of seawater involves salinity, so that an equation for salinity must be formulated to close the system. Very similarly to the conservation of heat :

$$\frac{\partial S}{\partial t} + (\mathbf{u}.\boldsymbol{\nabla})S = \nu_S \Delta S + \dot{S}$$

with $\nu_S \sim \nu_T/100$ the salt diffusivity of sea water, $\dot{S}$ (in ‰$/s$) sources and sinks of salt and S in ‰ or g/kg the concentration of dissolved salts.

- $\dot{S}$ represents air-sea water exchanges, river runoff and sea ice formation/fusion. Indeed, the salinity of sea ice is $S \sim 5$‰ so that its formation is a source of salt (brine rejection) for sea water.
- Salt diffusivity is by far lower than heat diffusivity, which can cause convective instabilities between water masses of different (θ, S) properties named salt fingering and convective layering. They have a relatively minor role for mixing and ocean circulation.

The Boussinesq equations

We have just derived a set of 7 equations with 8 unknowns :

$$
\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv &= -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + \nu(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})u \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}v + w\frac{\partial v}{\partial z} + fu &= -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + \nu(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})v \\
\frac{\partial P}{\partial z} = -\rho g^* \Longrightarrow P(z) &\simeq \rho_0 g^* \eta + g^* \int_z^0 \rho \, dz' \\
\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \Longrightarrow w(z) &= -\int_{-H}^{z} (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) dz' \\
\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} + w\frac{\partial \theta}{\partial z} &= \nu_T(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})\theta + \frac{1}{\rho_0 c_w}\dot{\Theta} \\
\frac{\partial S}{\partial t} + u\frac{\partial S}{\partial x} + v\frac{\partial S}{\partial y} + w\frac{\partial S}{\partial z} &= \nu_S(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})S + \dot{S} \\
\rho &= \rho(\theta, S, P_0(z))
\end{aligned}
$$

with $H > 0$ and η the ocean bottom depth and surface height.

The Boussinesq equations

Dynamic sea level has appeared as a new unknown in the vertical integration of the hydrostatic relation, hence a specific equation must be derived. The sea level obeys to

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Dynamic sea level has appeared as a new unknown in the vertical integration of the hydrostatic relation, hence a specific equation must be derived. The sea level obeys to the vertically-integrated continuity equation which also requires a surface and bottom kinematic boundary condition.

- Surface kinematic boundary condition :

The Boussinesq equations

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- Surface kinematic boundary condition :

$$\frac{d}{dt}(\eta - z)_\eta = P + R - E \Longrightarrow \frac{\partial \eta}{\partial t} = -\mathbf{u_h}.\boldsymbol{\nabla}_{\boldsymbol{h}}\eta + w(\eta) + P + R - E$$

- Bottom kinematic boundary condition :

The Boussinesq equations

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- Bottom kinematic boundary condition :

$$\frac{d}{dt}(z + H)_{-H} = 0 \Longrightarrow w(-H) = \frac{d}{dt}(-H) = -\mathbf{u_h}.\boldsymbol{\nabla}_{\boldsymbol{h}}H$$

- Vertically-integrated continuity :

The Boussinesq equations

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$$\frac{d}{dt}(\eta - z)_\eta = P + R - E \Longrightarrow \frac{\partial \eta}{\partial t} = -\mathbf{u_h}.\boldsymbol{\nabla_h}\eta + w(\eta) + P + R - E$$

- Bottom kinematic boundary condition :

$$\frac{d}{dt}(z + H)_{-H} = 0 \Longrightarrow w(-H) = \frac{d}{dt}(-H) = -\mathbf{u_h}.\boldsymbol{\nabla_h}H$$

- Vertically-integrated continuity :

$$\int_{-H}^{\eta} \frac{\partial w}{\partial z} dz = w(\eta) - w(-H) = \frac{\partial \eta}{\partial t} + \mathbf{u_h}(\eta).\boldsymbol{\nabla_h}\eta - \mathbf{u_h}(-H).\boldsymbol{\nabla_h}(-H)$$

The Boussinesq equations

where we have used Leibnitz's integration formula :

$$\int_{-H}^{\eta} -\boldsymbol{\nabla}_{\boldsymbol{h}}.\mathbf{u_h} dz = -\boldsymbol{\nabla}_{\boldsymbol{h}}.\int_{-H}^{\eta} \mathbf{u_h} dz + \mathbf{u_h}(\eta).\boldsymbol{\nabla}_{\boldsymbol{h}}\eta - \mathbf{u_h}(-H).\boldsymbol{\nabla}_{\boldsymbol{h}}(-H)$$

Finally :

$$\frac{\partial \eta}{\partial t} = -\boldsymbol{\nabla}_{\boldsymbol{h}}.\int_{-H}^{\eta} \mathbf{u_h} dz + P + R - E$$

- Hence the dynamic sea level is set by surface water exchanges and by vertically-integrated horizontal convergence. This is the 8th and last equation of the Boussinesq equation system.
- The Boussinesq approximations have permitted to filter out sound waves whose very large velocities $c_s \simeq 1500 m/s$ would have been a major issue for the numerical resolution of oceanic circulation. However, it still includes one type of fast waves that will require specific numerical treatments : external gravity waves with $c_g = \sqrt{gH} \sim 200 m/s$.

The Reynolds-Averaged Boussinesq equations

Principle :

- Ocean modelling does not resolve all the scales of motions, from the global scale to the millimetric scale of diffusion : $Re = \frac{UL}{\nu} \sim 10^{11}$, hence 10^4 moles of grid points would be needed !
- Hence a formal separation is needed to identify the influence of small-scale unresolved motion on the large-scale resolved motion.
- A Reynolds decomposition separates all variables into a mean and a perturbation, e.g. $\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}'$, with the mean being an ensemble average.
- Strong hypothesis of numerical modelling : ergodic hypothesis which assimilates ensemble to spatio-temporal means, so that $\overline{\mathbf{u}}$ is the large-scale (resolved) variable and $\mathbf{u}'$ is the small-scale (unresolved and to be parametrized) variable.

The Reynolds-Averaged Boussinesq equations

Under Reynolds's hypotheses (linearity, commutativity and indempotency), all non-linear terms of the Boussinesq equations are modified, the linear ones remaining unchanged.

- The time derivative is linear :

The Reynolds-Averaged Boussinesq equations

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- The time derivative is linear :

$$\overline{\frac{\partial u}{\partial t}} = \overline{\frac{\partial(\overline{u} + u')}{\partial t}} = \overline{\frac{\partial \overline{u}}{\partial t} + \frac{\partial u'}{\partial t}} = \frac{\partial \overline{\overline{u}}}{\partial t} + \frac{\partial \overline{u'}}{\partial t} = \frac{\partial \overline{u}}{\partial t}$$

- Meridional advection is nonlinear (second-order) :

The Reynolds-Averaged Boussinesq equations

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- Meridional advection is nonlinear (second-order) :

$$\begin{aligned} \overline{v\frac{\partial u}{\partial y}} &= \overline{(\overline{v}+v')\frac{\partial(\overline{u}+u')}{\partial y}} = \overline{\overline{v}\frac{\partial \overline{u}}{\partial y}} + \overline{\overline{v}\frac{\partial u'}{\partial y}} + \overline{v'\frac{\partial \overline{u}}{\partial y}} + \overline{v'\frac{\partial u'}{\partial y}} \\ &= \overline{v}\frac{\partial \overline{u}}{\partial y} + \overline{v}\frac{\partial \overline{u'}}{\partial y} + \overline{v'}\frac{\partial \overline{u}}{\partial y} + \overline{v'\frac{\partial u'}{\partial y}} = \overline{v}\frac{\partial \overline{u}}{\partial y} + \overline{v'\frac{\partial u'}{\partial y}} \end{aligned}$$

The Reynolds-Averaged Boussinesq equations

Similar result for zonal and vertical advection, so that with continuity :

The Reynolds-Averaged Boussinesq equations

Similar result for zonal and vertical advection, so that with continuity :

$$\overline{(\mathbf{u}.\boldsymbol{\nabla})u} = (\overline{\mathbf{u}}.\boldsymbol{\nabla})\overline{u} + \overline{(\mathbf{u}'.\boldsymbol{\nabla})u'} = (\overline{\mathbf{u}}.\boldsymbol{\nabla})\overline{u} + \boldsymbol{\nabla}.(\overline{\mathbf{u}'u'})$$

The second term is a turbulent (or eddy) transport contribution in the equation for the Reynolds-averaged zonal momentum $\overline{u}$. The covariance of zonal velocity with each component of velocity at the turbulent (hence unresolved) scale impacts the mean (resolved) momentum equation.

The Reynolds-Averaged Boussinesq equations

Similarly for meridional velocity and for tracers θ :

The Reynolds-Averaged Boussinesq equations

Similarly for meridional velocity and for tracers θ :

$$\begin{aligned}
\overline{(\mathbf{u}.\boldsymbol{\nabla})v} &= (\overline{\mathbf{u}}.\boldsymbol{\nabla})\overline{v} + \boldsymbol{\nabla}.(\overline{\mathbf{u}'v'}) \\
\overline{(\mathbf{u}.\boldsymbol{\nabla})\theta} &= (\overline{\mathbf{u}}.\boldsymbol{\nabla})\overline{\theta} + \boldsymbol{\nabla}.(\overline{\mathbf{u}'\theta'}) \\
\overline{(\mathbf{u}.\boldsymbol{\nabla})S} &= (\overline{\mathbf{u}}.\boldsymbol{\nabla})\overline{S} + \boldsymbol{\nabla}.(\overline{\mathbf{u}'S'})
\end{aligned}$$

12 additional transport terms appear in the conservation of horizontal momentum, heat and salt. Hence 12 new equations are needed to close the Boussinesq system.

The Reynolds-Averaged Boussinesq equations

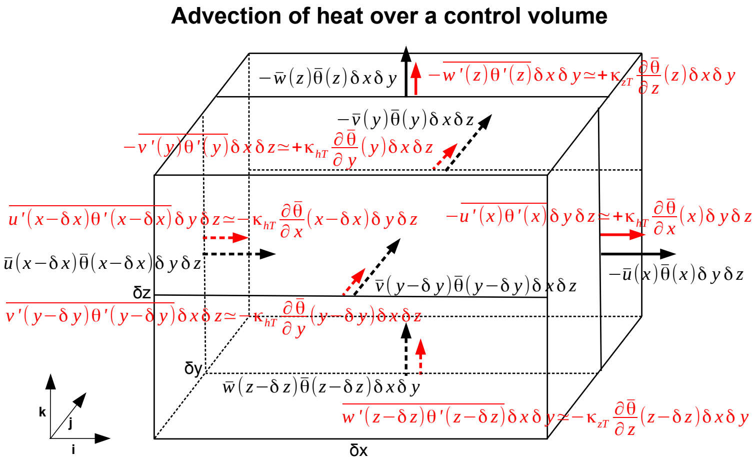

Figure 7 – Heat advection over the control volume : mean (resolved, black) and turbulent (unresolved, red), the latter being parametrized as a so-called "turbulent diffusion".

The Reynolds-Averaged Boussinesq equations

Closure hypothesis : introduction of turbulent diffusivities :

- We assume just like molecular diffusion the flux-gradient relation so that each turbulent flux $\overline{\mathbf{u}'X'}$ (with X either u, v, θ or S) is proportional to the gradient of the Reynolds-averaged (resolved) quantity $\nabla\overline{X}$.
- More specifically, we separate vertical and horizontal eddy fluxes, the former being damped by gravity, and we pose :

The Reynolds-Averaged Boussinesq equations

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- More specifically, we separate vertical and horizontal eddy fluxes, the former being damped by gravity, and we pose :

$$\overline{\mathbf{u_h}'X'} = -\kappa_{hX}\boldsymbol{\nabla}_{\boldsymbol{h}}\overline{X}, \quad \overline{w'X'} = -\kappa_{zX}\frac{\partial\overline{X}}{\partial z}$$

with κ_{hX} and κ_{zX} the horizontal and vertical eddy diffusivities for the variable X.

- Those diffusivities are several orders of magnitude larger than the molecular diffusivities in the momentum, temperature and salinity equations.

The Reynolds-Averaged Boussinesq equations

Finally :

$$
\begin{aligned}
\frac{\partial \overline{\mathbf{u_h}}}{\partial t} + (\overline{\mathbf{u}}.\boldsymbol{\nabla})\overline{\mathbf{u_h}} + f\mathbf{k}\times\overline{\mathbf{u_h}} &= -\frac{1}{\rho_0}\boldsymbol{\nabla}_{\boldsymbol{h}}\overline{P} + \boldsymbol{\nabla}_{\boldsymbol{h}}.(\kappa_{hu}\boldsymbol{\nabla}_{\boldsymbol{h}})\overline{\mathbf{u_h}} + \frac{\partial}{\partial z}(\kappa_{zu}\frac{\partial \overline{\mathbf{u_h}}}{\partial z})\\
\overline{P}(z) &= \rho_0 g\overline{\eta} + g\int_z^0 \overline{\rho}\, dz'\\
\overline{w}(z) &= -\int_{-H}^{z}\boldsymbol{\nabla}_{\boldsymbol{h}}.\overline{\mathbf{u_h}}\, dz'\\
\frac{\partial\overline{\eta}}{\partial t} &= -\boldsymbol{\nabla}_{\boldsymbol{h}}.\int_{-H}^{\eta}\overline{\mathbf{u_h}}\, dz + \overline{P} + \overline{R} - \overline{E}\\
\frac{\partial\overline{\theta}}{\partial t} + (\overline{\mathbf{u}}.\boldsymbol{\nabla})\overline{\theta} &= \boldsymbol{\nabla}_{\boldsymbol{h}}.(\kappa_{hT}\boldsymbol{\nabla}_{\boldsymbol{h}})\overline{\theta} + \frac{\partial}{\partial z}(\kappa_{zT}\frac{\partial\overline{\theta}}{\partial z}) + \frac{1}{\rho c_w}\overline{\dot{\Theta}}\\
\frac{\partial\overline{S}}{\partial t} + (\overline{\mathbf{u}}.\boldsymbol{\nabla})\overline{S} &= \boldsymbol{\nabla}_{\boldsymbol{h}}.(\kappa_{hS}\boldsymbol{\nabla}_{\boldsymbol{h}})\overline{S} + \frac{\partial}{\partial z}(\kappa_{zS}\frac{\partial\overline{S}}{\partial z}) + \overline{\dot{S}}\\
\overline{\rho} &= \overline{\rho}(\overline{\theta},\overline{S},\overline{P_0}(z))
\end{aligned}
$$

The Reynolds-Averaged Boussinesq equations

The Reynolds average sign above all variables of the equation system reminds us that it is far from describing the "truth" of ocean circulation. We assumed that :

- Turbulence, which is an advective process, can be modelled as a diffusive process. In particular, we assume that turbulent fluxes are a function of the local large-scale variables (locality), that they are proportional to their gradients (flux-gradient relation), that they only flux the properties down this gradient (downgradient fluxes).
- A deterministic relation exists between turbulent fluxes and the averaged quantities, although turbulent motion is chaotic and hence largely random by nature.
- BUT turbulent diffusivities are generally not constant and can have a complex mathematical formulation, in order to mimic the wealth of unresolved turbulent processes (e.g. convection, shear instabilities, wave breaking, etc.)

Dimensional analysis

The full equations of motion are in a Cartesian coordinates frame :

$$
\begin{aligned}
\frac{Du}{Dt} - \frac{uv\tan(\phi)}{a} + \frac{uw}{a} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v\sin(\phi) - 2\Omega w\cos(\phi) + \nu\Delta u \\
\frac{Dv}{Dt} - \frac{u^2\tan(\phi)}{a} + \frac{vw}{a} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u\sin(\phi) + \nu\Delta v \\
\frac{Dw}{Dt} - \frac{u^2+v^2}{a} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + 2\Omega u\cos(\phi) - g^* + \nu\Delta w
\end{aligned}
$$

The additional terms on the left-hand side correspond to the metric terms, and the additional terms involving Ω in the right-hand side are the horizontal Coriolis acceleration.

Dimensional analysis

We are interested in large-scale motions of typical scales :

Dimensional analysis

We are interested in large-scale motions of typical scales :

- $L \sim 1000km$
- $H \sim 1000m$
- $U \sim 0.1m/s$
- We deduce from the continuity equation the typical vertical velocity scale : $W \sim \frac{H}{L}U = 0.1mm/s$
- The characteristic timescale of those motions is hence : $T \sim \frac{L}{U} = 10^7 s \sim 1$ year
- At mid-latitude $f_0 = 2\Omega sin(\phi) \simeq 2\Omega cos(\phi) \simeq 10^{-4}s^{-1}$
- The water kinematic viscosity scales as $\nu \sim 10^{-3}m^2/s$

Dimensional analysis

Zonal	$\frac{Du}{Dt}$	$-\frac{uv\tan(\phi)}{a}$	$+\frac{uw}{a}$	$=$	$-\frac{1}{\rho_0}\frac{\partial p}{\partial x}$	$+\frac{\rho'}{\rho_0^2}\frac{\partial p}{\partial x}$	$+2\Omega v\sin(\phi)$	$-2\Omega w\cos(\phi)$	$+\nu\Delta u$
Meridional	$\frac{Dv}{Dt}$	$-\frac{u^2\tan(\phi)}{a}$	$+\frac{vw}{a}$	$=$	$-\frac{1}{\rho_0}\frac{\partial p}{\partial y}$	$+\frac{\rho'}{\rho_0^2}\frac{\partial p}{\partial y}$	$-2\Omega u\sin(\phi)$		$+\nu\Delta v$
OoM	U^2/L	U^2/a	UW/a		$\delta P_L/(\rho_0 L)$	$(\delta P_L\rho')/(\rho_0^2 L)$	$f_0 U$	$f_0 W$	$\nu U/H^2$
Value	10^{-8}	10^{-9}	10^{-12}		$? = 10^{-5}$	$?/1000 = 10^{-8}$	10^{-5}	10^{-8}	10^{-10}

Table 1 – Orders of magnitude (OoM) for large-scale horizontal motion.

Dominating balance :

Dimensional analysis

Zonal	$\frac{Du}{Dt}$	$-\frac{uv\tan(\phi)}{a}$	$+\frac{uw}{a}$	$=$	$-\frac{1}{\rho_0}\frac{\partial p}{\partial x}$	$+\frac{\rho'}{\rho_0^2}\frac{\partial p}{\partial x}$	$+2\Omega v\sin(\phi)$	$-2\Omega w\cos(\phi)$	$+\nu\Delta u$
Meridional	$\frac{Dv}{Dt}$	$-\frac{u^2\tan(\phi)}{a}$	$+\frac{vw}{a}$	$=$	$-\frac{1}{\rho_0}\frac{\partial p}{\partial y}$	$+\frac{\rho'}{\rho_0^2}\frac{\partial p}{\partial y}$	$-2\Omega u\sin(\phi)$		$+\nu\Delta v$
OoM	U^2/L	U^2/a	UW/a		$\delta P_L/(\rho_0 L)$	$(\delta P_L\rho')/(\rho_0^2 L)$	$f_0 U$	$f_0 W$	$\nu U/H^2$
Value	10^{-8}	10^{-9}	10^{-12}		$? = 10^{-5}$	$?/1000 = 10^{-8}$	10^{-5}	10^{-8}	10^{-10}

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Dominating balance : geostrophy. The next order terms :

Dimensional analysis

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Meridional	$\frac{Dv}{Dt}$	$-\frac{u^2\tan(\phi)}{a}$	$+\frac{vw}{a}$	$=$	$-\frac{1}{\rho_0}\frac{\partial p}{\partial y}$	$+\frac{\rho'}{\rho_0^2}\frac{\partial p}{\partial y}$	$-2\Omega u\sin(\phi)$		$+\nu\Delta v$
OoM	U^2/L	U^2/a	UW/a		$\delta P_L/(\rho_0 L)$	$(\delta P_L\rho')/(\rho_0^2 L)$	f_0U	f_0W	$\nu U/H^2$
Value	10^{-8}	10^{-9}	10^{-12}		$? = 10^{-5}$	$?/1000 = 10^{-8}$	10^{-5}	10^{-8}	10^{-10}

Table 1 – Orders of magnitude (OoM) for large-scale horizontal motion.

Dominating balance : geostrophy. The next order terms :

- The momentum trend and advection ; however turbulent advection can be strong enough to become a leading-order term in the surface layer (Ekman or convective layer) when $\kappa_{zu}U/H^2 \to 10^{-5}$;
- The non-Boussinesq contribution to the horizontal pressure gradient ;
- The horizontal Coriolis acceleration ;
- The main metric terms ;
- The vertical molecular viscosity ;
- The secondary metric terms ;
- The horizontal molecular viscosity, despite being the ultimate kinetic energy sink !

Dimensional analysis

Vertical	$\frac{Dw}{Dt}$	$-\frac{u^2+v^2}{a}$	$=$	$-\frac{1}{\rho_0}\frac{\partial p}{\partial z}$	$+\frac{\rho'}{\rho_0^2}\frac{\partial p}{\partial z}$	$+2\Omega u\cos(\phi)$	$-g^*\frac{\rho_0}{\rho}$	$-g^*\frac{\rho'}{\rho}$	$+\nu\Delta w$
OoM	UW/L	U^2/a		$\delta P_H/(\rho_0 H)$	$\delta P_H\rho'/(\rho_0^2 H)$	$f_0 U$	g^*	$g^*\rho'/rho_0$	$\nu W/H^2$
Value	10^{-11}	10^{-9}		$? = 10$	$?/1000 = 10^{-2}$	10^{-5}	10	10^{-2}	10^{-13}

Table 2 – Orders of magnitude (OoM) for large-scale vertical motion.

Dominating balance :

Dimensional analysis

Vertical	$\frac{Dw}{Dt}$	$-\frac{u^2+v^2}{a}$	$=$	$-\frac{1}{\rho_0}\frac{\partial p}{\partial z}$	$+\frac{\rho'}{\rho_0^2}\frac{\partial p}{\partial z}$	$+2\Omega u\cos(\phi)$	$-g^*\frac{\rho_0}{\rho}$	$-g^*\frac{\rho'}{\rho}$	$+\nu\Delta w$
OoM	UW/L	U^2/a		$\delta P_H/(\rho_0 H)$	$\delta P_H \rho'/(\rho_0^2 H)$	$f_0 U$	g^*	$g^*\rho'/rho_0$	$\nu W/H^2$
Value	10^{-11}	10^{-9}		$? = 10$	$?/1000 = 10^{-2}$	10^{-5}	10	10^{-2}	10^{-13}

Table 2 – Orders of magnitude (OoM) for large-scale vertical motion.

Dominating balance : hydrostatism. Even larger domain of validity than geostrophy, which is why most ocean models are hydrostatic.

Vertical acceleration becomes significant in the perturbation analysis when $W \sim U \sim 0.1 m/s$ and $L \sim 1m$, that is for fully developed 3-dimensional turbulence. This is the case of convection which will thus have to be parametrized in the eddy diffusivity coefficients κ_{zu}, κ_{zT} and κ_{zS}.

Application : transport reconstruction from thermal wind

Using the dynamical method with a level of no motion at 4000m and a linear equation of state $\rho = \rho_0(-\alpha_\theta \theta + \beta_S S)$, *estimate the mean surface velocity and the integral transport across the Drake passage.* $\alpha_\theta \simeq 10^{-4}\ {}^\circ C^{-1}$, $\beta_S \simeq 10^{-3}\,‰^{-1}$, $f_0 \simeq -1 \times 10^{-4} s^{-1}$.

Hydrographic section across the Drake Passage

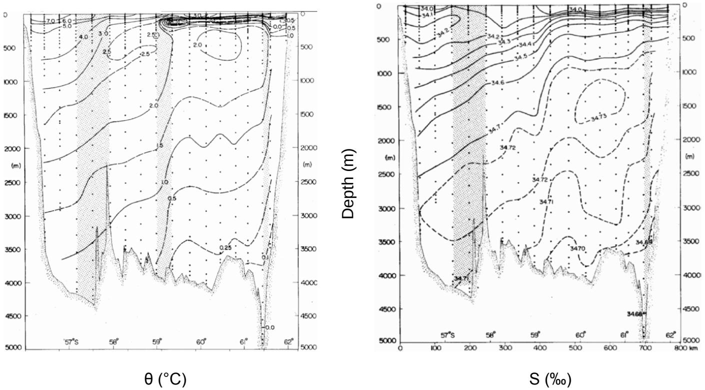

Figure 8 – Hydrographic section across Drake Passage (H. Johnson's lecture).

Application : transport reconstruction from thermal wind

Solution : the dynamical method consists in retrieving geostrophic velocities from the vertical integration of the thermal wind relation, which uses both geostrophy and hydrostatism, from a reference level. The thermal wind relation writes as :

$$
\begin{aligned}
\frac{\partial u_g}{\partial z} &= \frac{\partial}{\partial z}\left(-\frac{1}{f_0\rho_0}\frac{\partial P}{\partial y}\right) \\
&= -\frac{1}{f_0\rho_0}\frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z}\right) \\
&= -\frac{1}{f_0\rho_0}\frac{\partial}{\partial y}(-\rho g) \\
&= +\frac{g}{f_0\rho_0}\frac{\partial \rho}{\partial y} \\
&= +\frac{g}{f_0}\left(-\alpha_\theta\frac{\partial \theta}{\partial y}+\beta_S\frac{\partial S}{\partial y}\right)
\end{aligned}
$$

Application : transport reconstruction from thermal wind

We integrate it per layer of $\Delta z \sim 1000m$ height and over $\Delta y \simeq 500km$:

$$\begin{aligned}
\Delta\theta_{0-1000m} &\simeq +5^\circ C, \Delta S_{0-1000m} \simeq -0.2 \\
\Delta\theta_{1000-2000m} &\simeq +1.5^\circ C, \Delta S_{1000-2000m} \simeq -0.2 \\
\Delta\theta_{2000-3000m} &\simeq +1.5^\circ C, \Delta S_{2000-3000m} \simeq -0.02 \\
\Delta\theta_{3000-4000m} &\simeq +1^\circ C, \Delta S_{3000-4000m} \simeq +0.03
\end{aligned}$$

Hence we have :

$$\begin{aligned}
\Delta u_{3000-4000m} &= u_{3000m} = \frac{g\Delta z}{f_0 \Delta y}(-\alpha_\theta \Delta\theta_{3000-4000m} + \beta_S \Delta S_{3000-4000m}) \\
&\simeq -500 \times (-1 \times 10^{-4} + 0.03 \times 10^{-3}) \simeq +3.5cm/s \\
u_{2000m} &= u_{3000m} + \Delta u_{2000-3000m} \\
&\simeq 0.035 - 500 \times (-1.5 \times 10^{-4} - 0.02 \times 10^{-3}) \simeq 12cm/s \\
u_{1000m} &= u_{2000m} + \Delta u_{1000-2000m} \\
&\simeq 0.12 - 500 \times (-1.5 \times 10^{-4} - 0.2 \times 10^{-3}) \simeq 29.5cm/s \\
u_{0m} &= u_{1000m} + \Delta u_{0-1000m} \\
&\simeq 0.295 - 500 \times (-5 \times 10^{-4} - 0.2 \times 10^{-3}) \simeq 64.5cm/s
\end{aligned}$$

Application : transport reconstruction from thermal wind

We can deduce the integral transport across Drake passage by integrating meridionally and vertically those velocities :

$$
\begin{aligned}
T_{Drake} &\simeq \Delta y \Delta z (u_{4000m}/2 + u_{3000m} + u_{2000m} + u_{1000m} + u_{0m}/2) \\
&\simeq 5 \times 10^8 (0.035 + 0.12 + 0.295 + 0.645/2) \simeq 386 Sv
\end{aligned}
$$

This is the right order of magnitude for transports across the Drake Passage, although due to the numerical approximations it is overestimated by a factor $\sim 2-3$*. Those surface velocities and transports are among the most intense geostrophic currents found in the global ocean.*

Outline

The Equations of Ocean Circulation

Ocean modelling

An ocean model resolves numerically the equations of motion with the following specificities :

An ocean model resolves numerically the equations of motion with the following specificities :

- The coordinate system is not Cartesian but curvilinear (spherical) with horizontal axes that are not simply longitude and latitude.
- They are resolved over a finite number of grid cells and time steps, and hence must be discretized in time and space.
- A wide variety of lateral and vertical physical parametrizations can be introduced which all aim at modelling the unresolved turbulent motions.
- All boundary conditions (surface, lateral and bottom) must be specified for the equations to be solved in a given domain.

Discretization

Time discretization :

- The equations for u, v, η, θ and S are prognostic, which allows to step forward in time. They must be discretized in time.
- Example of time stepping scheme : the leapfrog scheme

$$X(t+\delta t) = X(t-\delta t) + 2\delta t RHS(t)$$

  with X any ocean prognostic variable, δt the timestep and $RHS(t)$ the right hand side of X evolution equation.
- Typical timesteps : $\sim 1h$ for ocean climate models to $\sim 10min$ for regional ocean models and $\sim 1min$ for coastal models.
- Truncation accuracy : behaviour of a scheme's error as a function of timestep δt (or grid spacing δx for space discretization). Example : the leapfrog has 2nd order accuracy (error of $O(\delta t^3)$).

Discretization

Space discretization : model grids

- Curvilinear grids : 3D arrays of points with orthogonal coordinates (i,j,k), one vertical and two horizontal.
- Horizontal directions do not strictly follow longitude and latitude, which allows to position the poles over continents.
- Consequences : modification of horizontal derivative operators, variable cell volume.

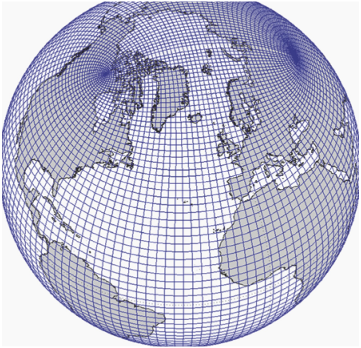

Figure 9 – NEMO model's tripolar curvilinear grid

Discretization

Horizontal discretization

- Horizontal resolution : $\delta x \sim 100km$ for global climate applications to $\sim 10km$ for ocean-only regional studies and $\sim 1km$ for coastal applications.
- Large variety of horizontal discretization scheme : either globally conserving for climate, or conserving local variance for small scales.

Discretization

Vertical discretization

- 3 main paradigms : truely vertical (z), terrain-following (σ) or isopycnal (ρ).
- Advantage of z-coordinate : natural to write horizontal momentum equations !

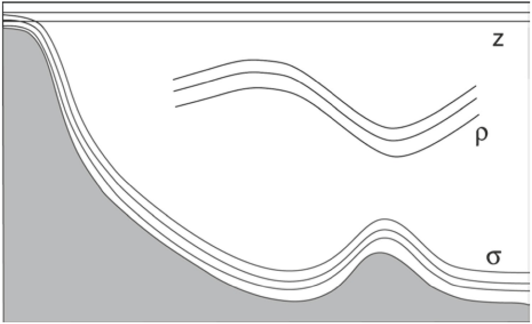

Figure 10 – Schematic of the three main ocean vertical coordinates in their natural domains of application : vertical z within the mixed layer, sigma σ at the bottom and isopycnal ρ within the interior (BM's lecture).

Discretization

Vertical discretization

- NEMO's z-coordinate has an irregular resolution, higher (typically $\delta z \sim 1 - 5m$) in the near-surface and lower (typically $\delta z \sim 100 - 300m$) at depth.
- Indeed vertical gradients are stronger near surface, which requires a higher resolution.
- This surface bias also relates to the larger interest in surface ocean for biological and weather/climate applications and by the lack of knowledge about the abyssal ocean.

Discretization

Location of variables at each grid cell :

- Most ocean models such as NEMO use the so-called Arakawa C-grid.
- This arrangement ensures important conservation properties for scalar variables.

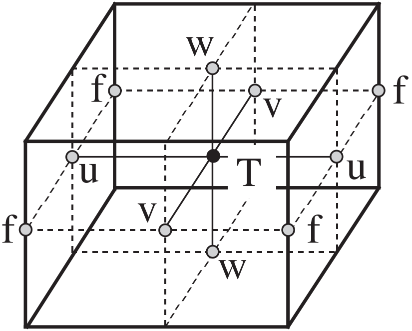

Figure 11 – *Schematic of NEMO's Arakawa-C grid (Madec et al 2016).*

Discretization

Relation between space and time resolution :

- "Courant-Friedrich-Lewy" (CFL) criterion : any information should not travel more than one grid cell in one timestep :

$$U\delta t < \delta x$$

  with U either the wave or advective velocity.
- Typical climate ocean model of timestep $\delta t \sim 1h$ and resolution $\delta x \sim 100km$:

Discretization

Relation between space and time resolution :

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$$U\delta t < \delta x$$

  with U either the wave or advective velocity.
- Typical climate ocean model of timestep $\delta t \sim 1h$ and resolution $\delta x \sim 100km$:

$$U < \frac{\delta x}{\delta t} \sim 20m/s$$

- External gravity waves of phase speed $c_g \sim 200m/s$ are allowed by the equation of sea level η and would cause numerical instabilities ! And they are not crucial to ocean circulation. Most common solution : filtering them out.

Lateral physics

General considerations :

- Lateral exchanges are enhanced in the ocean because no work is required against the buoyancy (or gravity) force.
- They are believed to be dominated by the stirring of oceanic properties by quasi-geostrophic mesoscale eddies.
- Having said that, it is fair to say that little is known about the actual level of horizontal mixing by mesoscale eddies and the value chosen by modellers responds to numerical stability constraints.
- Indeed, horizontal mixing operators and coefficients must be tuned to prevent any numerical instability to develop at the small scale, while at the same time not smooting too much the fine-scale oceanic structures.

Lateral physics

Lateral tracer physics :

- Mesoscale eddies are known to stir tracers (θ, S) along isopycnals, rather than along horizontal surface. The horizontal Laplacian operator is therefore slightly rotated to become isopycnal. Typically, $\kappa_{hT} \simeq 100m^2/s$ for a global model, but this value should decrease with increasing resolution as mesoscale eddies start being explicitely resolved.
- Mesoscale eddies restratify the ocean, which is not accounted for by a diffusive operator. Indeed, they are mostly formed by baroclinic instability extracts potential energy. A suitable parametrization of this effect is the addition of so-called "eddy-induced velocities" $\mathbf{u_{EIV}}$.

Lateral physics

Lateral momentum physics :

- Resolving small-scale dynamical structures is crucial because most of the ocean kinetic energy lies at the mesoscale.
- Hence a bilaplacian horizontal operator $\Delta_h^2 = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$ is preferred : it is more scale-selective, so that it permits smaller dynamical structures for a given resolution.

Lateral physics

Lateral boundary conditions :

- Tracers : to a very good approximation (neglecting geothermal fluxes), no lateral exchanges occur at the boundary with solid Earth.
- Momentum :
  - No normal flow at lateral boundaries
  - But the condition on tangent flow is more challenging to determine. A continuum of options between free-slip and no-slip.

Vertical physics

General considerations :

- At the small scale, turbulence occurs over all three directions of space, but because of gravity, the gradients of physical properties are mostly vertical.
- Hence parametrized with vertical diffusivity coefficients.
- The core of vertical physics in an ocean model is the parametrization of turbulence, a general theoretical framework giving the values of diffusivities as a function of the large-scale (resolved) structure of the flow.

Vertical physics

The Turbulent Kinetic Energy (TKE) scheme :

- Most used turbulence scheme in NEMO, although not the only one.
- Principle : resolve a simplified prognostic equation for the turbulent (unresolved) kinetic energy $\overline{\mathbf{u}'^2}$ and assume that vertical turbulent diffusivities scale with it.
- Relation TKE - diffusivities :

$$\begin{aligned} \kappa_{zu} &\propto l\sqrt{\overline{\mathbf{u}'^2}} \\ \kappa_{zT} = \kappa_{zS} &= \frac{\kappa_{zu}}{Pl} \end{aligned}$$

  with l a vertical mixing length scale and Pl the Prandtl number.
- A few comments on the prognostic TKE equation :
  - Vertical shear $\frac{\partial \mathbf{u_h}}{\partial z}$ is always a source of turbulence while vertical stratification $\frac{\partial \rho}{\partial z}$ can either be a source (if unstable) or a sink (if stable).
  - External gravity wave breaking is included as a surface source of turbulent kinetic energy, as are internal waves breaking in the mixed layer through an additional source distributed within that layer.

Vertical physics

The convection scheme :

- In the case of static instability, convection should efficiently mix water masses. However, convection is not allowed due the hydrostatic assumption.
- The Enhanced Vertical Diffusion (EVD) scheme is very simple : when static instability occurs

$$\kappa_{zT} = \kappa_{zS} = \kappa_{zu} \simeq 10m^2/s$$

so that within a few hours of simulation, stability is restored.

Vertical physics

In the interior ocean, where turbulence is weak and convection is absent :

- Diffusivities fall down to a background value, typically $\kappa_{zT} = \kappa_{zS} \simeq 10^{-5} m^2/s$ and $\kappa_{zu} \simeq 10^{-4} m^2/s$. They rather reflect the poor knowledge of oceanographers about mixing in the abyssal ocean.

Vertical physics

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- Internal wave-induced mixing distributes over the whole water column the mixing resulting from a climatology of internal wave dissipation energy. Its value typically does not exceed $0.01 m^2/s$, so that it is mostly active in the interior ocean and the stratified thermocline.
- Double diffusion mixing accounts for the instabilities caused by the different molecular diffusivities of heat and salt in sea water. They are the only ones accounting for differential mixing between salt and heat, so that $\kappa_{zT} \neq \kappa_{zS}$.

Vertical physics

Bottom boundary condition :

- Tracers : same as lateral boundaries, no flux.
- Momentum : bottom friction as a function of an internal wave (mostly tidal) dissipation climatology.

Surface forcing

Already described in the previous lecture. Specificities in ocean models :

- Only solar heat flux Q_{SW} and river runoff R (and ice shelves/iceberg melting in few configurations) are penetrative fluxes, the former with an exponential decay and the second applying evenly over typically $\sim 30m$.
- All other heat, water fluxes and the turbulent momentum flux (wind stress) only apply to the first model level ($\sim 1-5m$ thickness).
- Usually, vertical diffusivities are large in the first levels that define the mixed layer, so that those surface fluxes are in practise very rapidly redistributed over the mixed layer depth.
- But below typically $50m$ depth, the ocean does not feel directly surface fluxes.

Surface forcing

Consequence for the formulation of fluxes :

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$$-\rho_0 \overline{w'(0)\mathbf{u_h}'(0)} \simeq \rho_0 \kappa_{zu} \frac{\partial \mathbf{u_h}}{\partial z}\bigg|_0 \rightarrow \boldsymbol{\tau}_0 = \rho_a C_d |\mathbf{U(10m)}|\mathbf{U(10m)}$$

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- Heat and water forcings are external sources $\dot{\Theta}$ and $\dot{S}$ for temperature and salinity :

$$\begin{aligned}
\dot{\Theta}(z) &= \frac{1}{\rho_0 c_w \delta z} Q_{tot}(z) = \frac{1}{\rho_0 c_w \delta z}(Q_{SW}(z) + Q_{LW} + Q_S + Q_L) \\
\dot{S}(z) &= \frac{1}{\rho_0 \delta z}(E - P - R(z))
\end{aligned}$$

Surface forcing

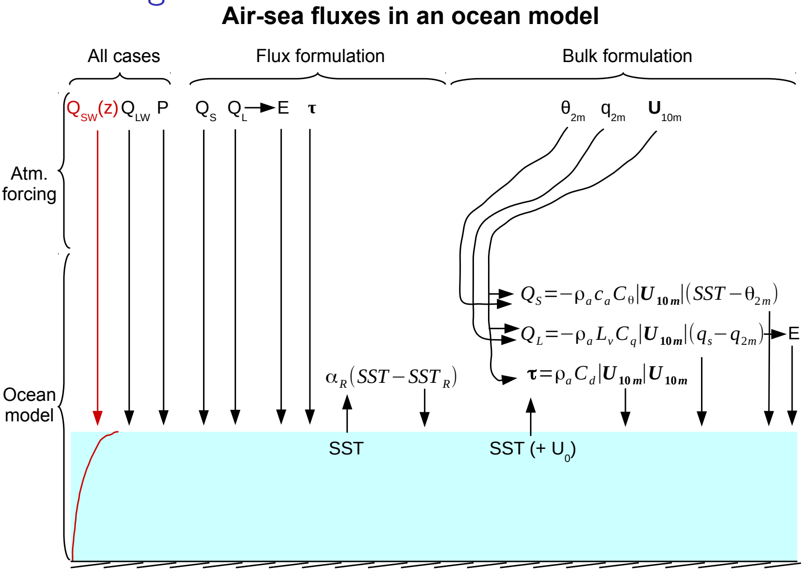

Figure 12 – Formulation of air-sea fluxes in an ocean model.

Surface forcing

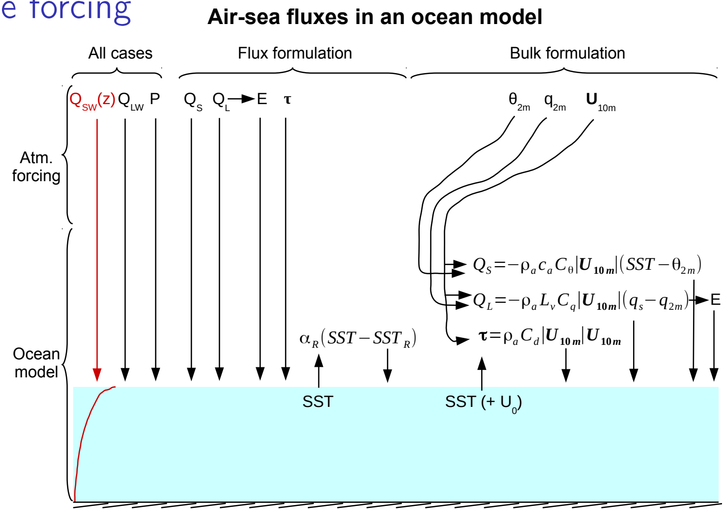

Figure 13 – Formulation of air-sea fluxes in an ocean model. Turbulent fluxes can either be directly taken from the forcing atmosphere ("flux method") or computed online within the oceanic model ("Bulk method") from the ocean and atmospheric surface parameters (temperature, humidity and wind).

Surface forcing

The special case of turbulent air-sea fluxes :

- In coupled mode, there is a continuous feedback between the ocean and atmosphere at the coupling frequency (typically a few hours), so that fluxes at the interface are consistent between both components.
- In the forced oceanic mode, turbulent fluxes can either be computed from surface atmospheric parameters (so-called "Bulk form") or taken as an external forcing from the atmosphere (so-called "flux form").
- A major issue with both strategies is that the ocean has more inertia than the atmosphere, so that the atmosphere should respond quickly to any air-sea flux, which is only possible in coupled mode. We have just stated that the forced oceanic configuration is an ill-defined problem compared to the forced atmospheric one.
- Hence even in the forced mode, some atmospheric feedback must be accounted for.

Surface forcing

The special case of turbulent air-sea fluxes :

- Advantage of the "Bulk form" : implicit feedback through sea surface temperature variations.
- Problems of the Bulk form :

Surface forcing

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- Advantage of the "Bulk form" : implicit feedback through sea surface temperature variations.
- Problems of the Bulk form :
  - Fluxes are computed from temporal averaged atmospheric parameters (typically a few hours), which can induce large errors because fluxes are nonlinear (the average flux is not the flux deduced from averaged parameters).
  - The "Bulk" formulas used might differ from those of the atmospheric forcing model, which causes an inconsistency between both models
- Trick with the "flux form" : adding a sea surface temperature (SST) restoration which mimics the coupling and ensures reasonable temperatures :

  $$Q_{tot} \rightarrow Q_{tot} + \alpha_r (SST - SST_{ref})$$

  with $\alpha_r \sim -50 W/m^2/^\circ C$ the restoration factor and SST_{ref} a reference SST.
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Surface forcing

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- Problem of the flux form : which SST_{ref} to use ?

Surface forcing

Conclusion : coupled ocean-atmosphere modelling is always preferable for oceanic applications !

Surface forcing

Historical versus hindcast modes :

- Hindcast mode : forced by an atmospheric hindcast, so that the historical chronology of past events can be reproduced by the ocean. Although a large part of ocean variability is also chaotic and not related to atmospheric forcing !
- Historical mode : uses a free atmospheric model only forced by historical anthropogenetic concentrations of greenhouse gases (and sometimes aerosols) as the forcing. The general warming trend can be reproduced, but no historical chronology is expected because no observation assimilated !

Model error and ensemble modelling

Sources of error inherent to ocean modelling :

Model error and ensemble modelling

Sources of error inherent to ocean modelling :

- Errors related to the approximations of the Boussinesq equation system ;
- Errors due to the closure of turbulence in the Reynolds-averaged framework ;
- Time and space numerical discretization errors ;
- Errors related to initial conditions and surface forcing.

The nonlinear nature of oceanic circulation predicts that any small error will tend to exponentially increase until reaching saturation : the "butterfly effect" !

Model error and ensemble modelling

Ensemble numerical modelling :

- Principle : sample the various sources of error (typically initial conditions, physics or atmospheric forcing).
- Applications :

Model error and ensemble modelling

Ensemble numerical modelling :

- Principle : sample the various sources of error (typically initial conditions, physics or atmospheric forcing).
- Applications :
  - Document or reduce errors in weather, ocean and climate predictions.
  - Study the chaotic part of ocean variability, which is not directly related to any forcing.
  - Interpret observations and evaluate more accurately ocean models.

A new paradigm for ocean modelling !

Example 1 : regional ocean model

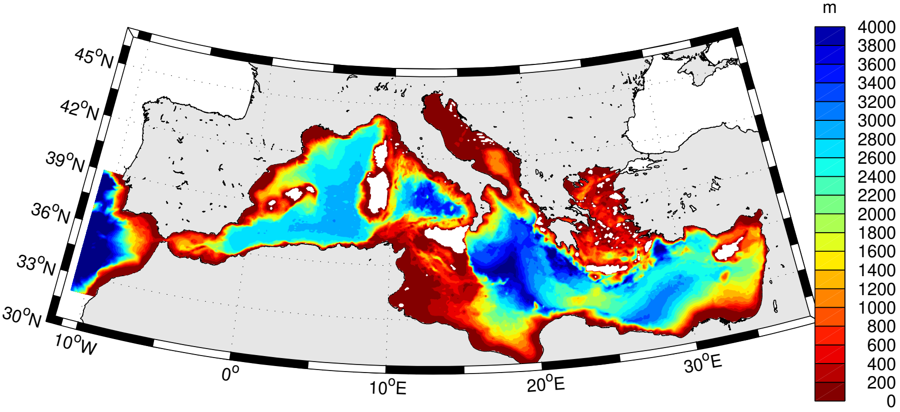

Figure 14 – *NEMOMED12 domain and bathymetry (Waldman et al 2017a).*

Example 1 : regional ocean model

Specificities of regional models :

- A higher resolution can be afforded :
  - Physics can be made eddy-resolving (resolving mesoscale eddies, that is $\sim 1/10^{\circ}$).
  - Bathymetry can better resolve channels, straits and interactions with topography.
  - Atmospheric forcing whose regional features can be made more accurate.
- Lateral boundary conditions at their open boundaries must be specified.

Example 1 : regional ocean model

NEMOMED12 model :

- Regional NEMO configuration on the Mediterranean Sea, a semi-enclosed mid-latitude sea.
- At its only open boundary with the global ocean, in the near-Gibraltar Atlantic Ocean, θ, S and η are restored towards an oceanic reanalysis and the domain is assumed to be closed.
- High-resolution regional modelling is required by its key exchanges at narrow straits, key high-resolution atmospheric jets and the need to resolve mesoscale dynamics. Its horizontal resolution is $1/12°$, that is $\sim 6-8km$, it is hence named an eddy-permiting model because it starts resolving mesoscale eddies.
- Vertical resolution ranges from 1m at surface to $\sim 100m$ at the bottom.
- Hindcast mode : the atmospheric flux forcing is a $12km$ resolution regional atmospheric reanalysis covering the period 1979–2013, meaning that observations are assimilated.
- Most of the physical options are identical to those presented before.
- Initial conditions are from an oceanic climatology.

Example 2 : global ocean climate model

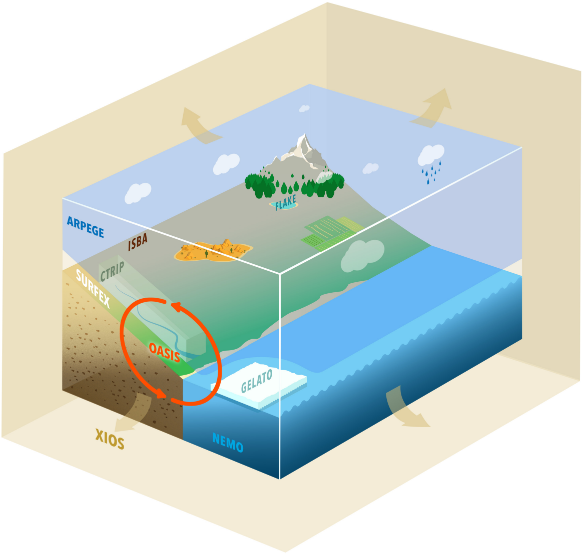

Figure 15 – *Schematic of CNRM-CM6 coupled model.*

Example 2 : global ocean climate model

CNRM-CM6 global coupled climate model :

- Participates in the next Climate Model Intercomparison Programme (CMIP6) in the framework of the International Panel on Climate Change (IPCC) sixth Assessment Report (AR6).
- Includes the main components of the climate system : ocean, sea ice, atmosphere, continental surfaces and atmospheric aerosols
- Horizontal resolutions are typically 1° for all components, the vertical oceanic resolution being identical to NEMOMED12 regional Mediterranean model.
- Historical mode : solar radiations, anthropogenetic greenhouse gases and aerosols (both natural and anthropogenetic) are the only time-varying external forcings.
- 10-member initial state ensemble to document the internal climate variability.
- Priorly equilibrated in a so-called pre-industrial control simulation.

Example 2 : global ocean climate model

Ocean component : NEMO 1°

- Physical parametrizations essentially identical to those of NEMOMED12. The only notable difference is the inclusion of the mesoscale eddy-induced velocity parametrization for tracers (temperature and salinity), because mesoscale eddies are not resolved.
- The ocean surface is fully coupled with the atmosphere at a 6-hourly frequency

Example 2 : global ocean climate model

Sea ice component : GELATO model embedded into NEMO

- It resolves both the sea ice and snow (above sea ice) dynamics and thermodynamics, including their exchanges with both the atmosphere and ocean.
- Prognostic variables are the sea ice and snow volume and enthalpy, the snow density and the sea ice surface, salinity and age.
- Over each oceanic grid cell, a fraction between 0 and 1 of sea ice area covered with snow is present.

Practical aspects of numerical modelling

Where do I read the physical description of my run ?

- The core of the model is written in its Fortran routines, and unless specific model development is required, no intervention is needed.
- Most of the options that users might want to modify are written in a so-called namelist, which is a file specifying the values for the corresponding parameters.
- A set of fundamental options must be specified as compliation keys, in a separate file. Those options are read during the model compilation so that once it is compiled no further change can be made on them.

Practical aspects of numerical modelling

Grid and mask variables :

- Longitudes, latitudes, depths, land-sea masks and scaling factors (δx, δy, δz) are variable at each grid point and differ between the T-grid and the grids for velocities (U-grid, V-grid and W-grid).
- An important consequence of this is that all space averages should be computed as ponderate means that account for each grid cell's volume, e.g. :

$$<\theta>=\frac{\Sigma\theta\delta x\delta y\delta z}{\Sigma\delta x\delta y\delta z}$$

  with $<\theta>$ an arbitrary 3D average.
- Another consequence, although of lesser importance, is that at a given location (i,j,k), the T-grid can be over the sea while the U-grid (or V-grid) is over land, or vice versa.

Practical aspects of numerical modelling

Online and offline diagnostics : the example of zonal temperature advection $u\frac{\partial \theta}{\partial x}$.

- Online diagnostic : storage of its contribution to the temperature trend during the model computation. Hence at the model time step, with the model mathematical formulation and numerical scheme for tracer advection.
- Offline diagnostic : trying to retrieve it after the run has already been performed, from the model outputs $\overline{u}$ and $\overline{\theta}$ which are generally stored every month or day. Errors : the computation is not done at the model time step, and the mathematical formulation and numerical scheme used might not be identical to those of the model.

Conclusion : online diagnostic is always preferable, but in practice some diagnostics have to be performed offline !